Duke University

Math 218D-2

Matrices and Vectors

Exam I

Name: Solutions

Unique ID:

I have adhered to the Duke Community Standard in completing this exam.

Signature:

May 30, 2024

- There are 100 points and 14 problems on this 100-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

Duke University MATH

(4 pts) **Problem 1.** Suppose $S = A + A^\intercal$ where A is 2024×2024. Show that S is symmetric. **Avoid circular logic to receive credit.**

Solution. $S^\intercal = (A + A^\intercal)^\intercal = A^\intercal + (A^\intercal)^\intercal = A^\intercal + A = S$

Problem 2. Consider $A = \begin{bmatrix} * & * & 0 & 6 & 3 & 12 \\ * & * & 0 & 2 & 1 & 4 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & 0 & 6 & 3 & 12 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & 0 & 4 & 2 & 8 \\ * & * & 0 & 2 & 1 & 4 \end{bmatrix}$. Entries in the first two columns are unknown and marked as $*$.

(3 pts) (a) A has __7__ rows and __6__ columns. The notation $\mathbb{R}^a \xrightarrow{A} \mathbb{R}^b$ is valid for $a =$ __6__ and $b =$ __7__.

(2 pts) (b) The matrix-vector products $A\boldsymbol{v}$ and $A^\intercal\boldsymbol{w}$ only make sense if $\boldsymbol{v}$ has __6__ coordinates and $\boldsymbol{w}$ has __7__ coordinates.

(2 pts) (c) Only one of the following linear combinations of the columns of A produces the zero vector. Select this linear combination.

√ $2\ \mathrm{Col}_3 + \mathrm{Col}_4 - 2\ \mathrm{Col}_5$ ◯ $\mathrm{Col}_4 + 2\ \mathrm{Col}_5$ ◯ $\mathrm{Col}_5 - 3\ \mathrm{Col}_6$ ◯ $\mathrm{Col}_3 - \mathrm{Col}_4$

(2 pts) (d) Each of the following options below describes a strategy for filling in the missing data in the first two columns of A. Select each strategy that would make A have "rank one" (each option is worth 0.5pts).

√ Set every $*$ equal to 0.

√ Set Col_1 equal to Col_4 and set Col_2 equal to Col_5.

◯ Set every $*$ equal to 1.

◯ Set every $*$ in Col_1 equal to 1 and every $*$ in Col_2 equal to 2.

(2 pts) (e) Suppose we set every $*$ equal to zero. Then which column is called the "first pivot column" of A?

◯ Col_1 ◯ Col_2 ◯ Col_3 √ Col_4 ◯ Col_5

(2 pts) **Problem 3.** Suppose that A and B are matrices and that $\boldsymbol{v}$ is a vector with the following properties.

$$v \in \mathbb{R}^8 \qquad A\boldsymbol{v} \in \mathbb{R}^5 \qquad B(A\boldsymbol{v}) \in \mathbb{R}^{13}$$

Then A is __5__ $\times$ __8__ and B is __13__ $\times$ __5__.

Problem 4. Suppose that $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathcal{E}_A(\lambda)$ and let $\boldsymbol{v}$ be defined by the linear combination $\boldsymbol{v} = c_1 \cdot \boldsymbol{v}_1 + c_2 \cdot \boldsymbol{v}_2$.

(2 pts) (a) Select all of the following symbols from this problem that represent a *scalar quantity* (no partial credit here).

○ $\boldsymbol{v}_1$ ○ $\boldsymbol{v}_2$ ○ A √ λ ○ $\boldsymbol{v}$ √ c_1 √ c_2

(2 pts) (b) The context of this problem requires which of the following adjectives to apply to A?

√ square ○ symmetric ○ rank one ○ identity ○ upper triangular

(4 pts) (c) Show that $\boldsymbol{v} \in \mathcal{E}_A(\lambda)$. **Avoid circular logic to receive credit.**

Solution. The data in the problem tells us that $A\boldsymbol{v}_1 = \lambda \cdot \boldsymbol{v}_1$ and that $A\boldsymbol{v}_2 = \lambda \cdot \boldsymbol{v}_2$. We want to demonstrate that $A\boldsymbol{v} = \lambda \cdot \boldsymbol{v}$. To do so, note that

$$\begin{aligned} A\boldsymbol{v} &= A(c_1 \cdot \boldsymbol{v}_1 + c_2 \cdot \boldsymbol{v}_2) \\ &= c_1 \cdot A\boldsymbol{v}_1 + c_2 \cdot A\boldsymbol{v}_2 \\ &= c_1 \cdot \lambda \cdot \boldsymbol{v}_1 + c_2 \cdot \lambda \cdot \boldsymbol{v}_2 \\ &= \lambda \cdot (c_1 \cdot \boldsymbol{v}_1 + c_2 \cdot \boldsymbol{v}_2) \\ &= \lambda \cdot \boldsymbol{v} \end{aligned}$$

Problem 5. The equations below depict the results of multiplying a matrix A by three vectors $\boldsymbol{w}$, $\boldsymbol{x}$, and $\boldsymbol{y}$.

$$\begin{bmatrix} & A & \end{bmatrix} \overset{\boldsymbol{w}}{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}} = \begin{bmatrix}5\\5\\5\\5\end{bmatrix} \qquad \begin{bmatrix} & A & \end{bmatrix} \overset{\boldsymbol{x}}{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}} = \begin{bmatrix}21\\9\\11\\7\end{bmatrix} \qquad \begin{bmatrix} & A & \end{bmatrix} \overset{\boldsymbol{y}}{\begin{bmatrix}0\\0\\0\\-1\end{bmatrix}} = \begin{bmatrix}25\\7\\12\\5\end{bmatrix}$$

(2 pts) (a) Which (if any) of the following vectors is an eigenvector of A? √ $\boldsymbol{w}$ ○ $\boldsymbol{x}$ ○ $\boldsymbol{y}$ ○ none of these

(4 pts) (b) Calculate $\text{Col}_1 + \text{Col}_3$, where Col_1 and Col_3 are the first and third columns of A. *Hint.* Start by explaining that this is the same thing as calculating $A\boldsymbol{v}$ for what vector $\boldsymbol{v}$?

Solution. This is the same as calculating $A\boldsymbol{v}$ for $\boldsymbol{v} = \begin{bmatrix}1 & 0 & 1 & 0\end{bmatrix}^\intercal = \boldsymbol{v} - \boldsymbol{x} + \boldsymbol{y}$. It follows that

$$\text{Col}_1 + \text{Col}_3 = A(\boldsymbol{v} - \boldsymbol{x} + \boldsymbol{y}) = A\boldsymbol{v} - A\boldsymbol{x} + A\boldsymbol{y} = \begin{bmatrix}5\\5\\5\\5\end{bmatrix} - \begin{bmatrix}21\\9\\11\\7\end{bmatrix} + \begin{bmatrix}25\\7\\12\\5\end{bmatrix} = \begin{bmatrix}9\\3\\6\\3\end{bmatrix}$$

(4 pts) **Problem 6.** Let G be the directed graph whose incidence matrix is given by

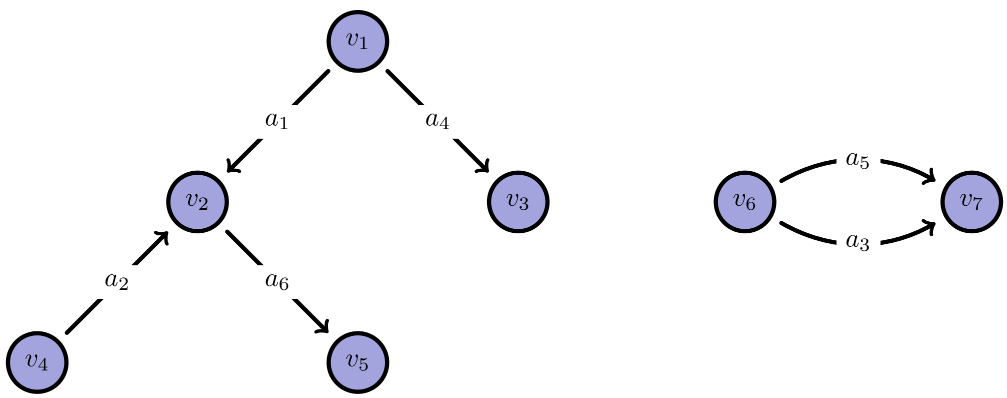
$$A = \begin{bmatrix} -1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Then $\chi(G) =$ __1__, the number of connected components of G is __2__, and the circuit rank of G is __1__.

Solution. The Euler characteristic is

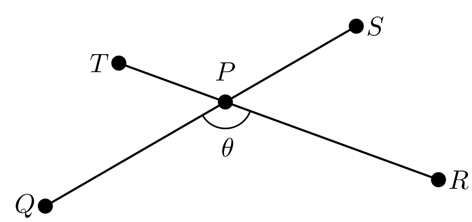
$$\chi(G) = (\text{number of nodes}) - (\text{number of arrows}) = (\text{number of rows of } A) - (\text{number of columns of } A) = 7 - 6 = 1$$

To calculate the number of connected components and the circuit rank, we need to draw the picture.



There are *two connected components*. Removing either a_3 or a_5 breaks all cycles, so the *circuit rank is one*.

Problem 7. This figure depicts five points P, Q, R, S, and T in $\mathbb{R}^n$. The figure shows that the line segment connecting Q to S intersects the line segment connecting R to T at the point P. The angle θ depicted is the angle between the line segment connecting P to Q and the line segment connecting P to R.



(2 pts) (a) Only one of the following vectors is equal to $\overrightarrow{QR}$. Select this vector.

√ $\overrightarrow{QP} + \overrightarrow{PR}$ ◯ $\overrightarrow{QP} - \overrightarrow{PR}$ ◯ $\overrightarrow{QP} + \overrightarrow{PS}$ ◯ $\overrightarrow{QP} + \overrightarrow{PQ}$

(2 pts) (b) Suppose that P and Q have coordinates $P(4, 5, 7, 10)$ and $Q(6, 6, 12, 11)$. Then $\|\overrightarrow{PQ}\| =$ __$\sqrt{31}$__.

Solution. The coordinates of $\overrightarrow{PQ}$ are $\overrightarrow{PQ} = \begin{bmatrix}6-4 & 6-5 & 12-7 & 11-10\end{bmatrix}^\intercal = \begin{bmatrix}2 & 1 & 5 & 1\end{bmatrix}^\intercal$. It follows that $\|\overrightarrow{PQ}\| = \sqrt{2^2 + 1^2 + 5^2 + 1^2} = \sqrt{31}$.

(2 pts) (c) Only one of the following values is equal to $\|\overrightarrow{RT}\|$. Select this value.

√ $\|\overrightarrow{TP}\| + \|\overrightarrow{RP}\|$ ◯ $\|\overrightarrow{RQ}\| + \|\overrightarrow{QT}\|$ ◯ $\|\overrightarrow{RP}\| + \|\overrightarrow{PS}\| + \|\overrightarrow{ST}\|$ ◯ $\|\overrightarrow{RP}\| - \|\overrightarrow{PT}\|$

(2 pts) (d) Suppose that $\langle \overrightarrow{PQ}, \overrightarrow{PR} \rangle = -5$. Then only one of the following statements about θ is correct. Select this statement.

◯ θ is an acute angle √ θ is an obtuse angle ◯ θ is a right angle

◯ we don't have enough information to conclude if θ is an acute, obtuse, or right angle

Problem 8. Suppose that A is a 2024×2023 matrix and that $\boldsymbol{v}$ and $\boldsymbol{w}$ are vectors in $\mathbb{R}^{2023}$ satisfying

$$\|\boldsymbol{v}\| = 5 \qquad \langle \boldsymbol{v}, \boldsymbol{w}\rangle = 5 \qquad \boldsymbol{v} \in \mathcal{E}_{A^\intercal A}(3) \qquad \boldsymbol{w} \in \mathcal{E}_{A^\intercal A}(0)$$

(3 pts) (a) Interpret the data in this problem by filling in the following blanks.

$$\langle \boldsymbol{v}, \boldsymbol{v}\rangle = \underline{\quad \|\boldsymbol{v}\|^2 = 25 \quad} \qquad A^\intercal A\boldsymbol{v} = \underline{\quad 3 \cdot \boldsymbol{v} \quad} \qquad A^\intercal A\boldsymbol{w} = \underline{\quad 0 \cdot \boldsymbol{w} = \boldsymbol{O} \quad}$$

(5 pts) (b) Calculate $\|A(\boldsymbol{v} - \boldsymbol{w})\|^2$. Clearly show your steps to receive credit.

Solution. The key is to expand $\|A(\boldsymbol{v} - \boldsymbol{w})\|^2$ using inner products and appeal to the adjoint formula.

$$\begin{aligned}
\|A(\boldsymbol{v} - \boldsymbol{w})\|^2 &= \langle A(\boldsymbol{v} - \boldsymbol{w}), A(\boldsymbol{v} - \boldsymbol{w})\rangle \\
&= \langle A^\intercal A(\boldsymbol{v} - \boldsymbol{w}), \boldsymbol{v} - \boldsymbol{w}\rangle \\
&= \langle A^\intercal A\boldsymbol{v} - A^\intercal A\boldsymbol{w}, \boldsymbol{v} - \boldsymbol{w}\rangle \\
&= \langle 3 \cdot \boldsymbol{v} - \boldsymbol{O}, \boldsymbol{v} - \boldsymbol{w}\rangle \\
&= \langle 3 \cdot \boldsymbol{v}, \boldsymbol{v} - \boldsymbol{w}\rangle \\
&= 3 \cdot \langle \boldsymbol{v}, \boldsymbol{v}\rangle - 3 \cdot \langle \boldsymbol{v}, \boldsymbol{w}\rangle \\
&= 3 \cdot 5^2 - 3 \cdot 5 \\
&= 3 \cdot 5 \cdot (5 - 1) \\
&= 15 \cdot 4 \\
&= 60
\end{aligned}$$

(5 pts) **Problem 9.** Consider a 2024×3 matrix A, so A looks like

$$A = \begin{bmatrix} | & | & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \\ | & | & | \end{bmatrix}$$

Suppose that each of the columns of A is a unit vector, that $\boldsymbol{a}_2$ is orthogonal to both $\boldsymbol{a}_1$ and $\boldsymbol{a}_3$, and that $\boldsymbol{a}_1$ forms an angle of $\theta = \pi/4$ with $\boldsymbol{a}_3$. Find the Gramian $A^\intercal A$ of A.

Hint. Recall that $\cos(\pi/4) = \sqrt{2}/2$.

Solution. The data from this problem tells us that

$$\langle \boldsymbol{a}_1, \boldsymbol{a}_1\rangle = \langle \boldsymbol{a}_2, \boldsymbol{a}_2\rangle = \langle \boldsymbol{a}_3, \boldsymbol{a}_3\rangle = 1 \qquad \langle \boldsymbol{a}_2, \boldsymbol{a}_1\rangle = \langle \boldsymbol{a}_2, \boldsymbol{a}_3\rangle = 0 \qquad \langle \boldsymbol{a}_1, \boldsymbol{a}_3\rangle = \|\boldsymbol{a}_1\| \cdot \|\boldsymbol{a}_3\| \cdot \pi/4 = 1 \cdot 1 \cdot \frac{\sqrt{2}}{2}$$

The Gramian of A is then

$$A^\intercal A = \begin{bmatrix} \langle \boldsymbol{a}_1, \boldsymbol{a}_1\rangle & \langle \boldsymbol{a}_1, \boldsymbol{a}_2\rangle & \langle \boldsymbol{a}_1, \boldsymbol{a}_3\rangle \\ \langle \boldsymbol{a}_2, \boldsymbol{a}_1\rangle & \langle \boldsymbol{a}_2, \boldsymbol{a}_2\rangle & \langle \boldsymbol{a}_2, \boldsymbol{a}_3\rangle \\ \langle \boldsymbol{a}_3, \boldsymbol{a}_1\rangle & \langle \boldsymbol{a}_3, \boldsymbol{a}_2\rangle & \langle \boldsymbol{a}_3, \boldsymbol{a}_3\rangle \end{bmatrix} = \begin{bmatrix} 1 & 0 & \sqrt{2}/2 \\ 0 & 1 & 0 \\ \sqrt{2}/2 & 0 & 1 \end{bmatrix}$$

Problem 10. Consider the calculation of the reduced row echelon form of the following 7×8 matrix A.

$$\operatorname{rref}\overset{A}{\begin{bmatrix} -2 & -10 & -1 & 1 & -11 & 0 & -2 & -7 \\ -3 & -15 & -2 & 2 & -18 & 0 & -3 & -12 \\ 1 & 5 & 1 & 0 & 4 & -1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & -1 \\ -3 & -15 & -3 & 3 & -21 & 0 & -2 & -16 \\ -2 & -10 & -1 & 1 & -11 & 0 & -1 & -8 \\ 1 & 5 & 1 & -2 & 10 & 1 & 0 & 9 \end{bmatrix}} = \begin{bmatrix} 1 & 5 & 0 & 0 & 4 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -3 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(4 pts) (*a*) $\operatorname{rank}(A) =$ __5__, $\operatorname{nullity}(A) =$ __3__, $\operatorname{rank}(A^\intercal) =$ __5__ and $\operatorname{nullity}(A^\intercal) =$ __2__

(2 pts) (*b*) Which columns of A are the *nonpivot columns*? Select all that apply (no partial credit here).

○ Col_1 √ Col_2 ○ Col_3 ○ Col_4 √ Col_5 ○ Col_6 ○ Col_7 √ Col_8

(2 pts) (*c*) Suppose $\boldsymbol{b} \in \mathbb{R}^7$ makes the system $A\boldsymbol{x} = \boldsymbol{b}$ consistent. Which of the variables in this system is *dependent*? Select all that apply (no partial credit here).

√ x_1 ○ x_2 √ x_3 √ x_4 ○ x_5 √ x_6 √ x_7 ○ x_8

(2 pts) (*d*) Which of the following adjectives correctly describes A? Select all that apply (no partial credit here).

○ full column rank ○ full row rank ○ full rank √ rank deficient ○ nonsingular ○ singular

(2 pts) (*e*) The columns of A satisfy all of the following equations. However, according to our terminology from class, only one of the following equations is called a *column relation*. Select this equation.

√ $\text{Col}_5 = 4\ \text{Col}_1 - 3\ \text{Col}_4$ ○ $\text{Col}_5 = -\text{Col}_1 + \text{Col}_2 - 3\ \text{Col}_4$

○ $\text{Col}_1 = -3\ \text{Col}_2 + 12\ \text{Col}_4 + 4\ \text{Col}_5$ ○ $\text{Col}_2 = -3\,\text{Col}_1 + 6\,\text{Col}_4 + 2\ \text{Col}_5$

Problem 11. Let R be a 5×6 reduced row echelon form matrix with exactly two column relations given by

$$\text{Col}_4 = -5\ \text{Col}_1 + 2\ \text{Col}_3 \qquad \text{Col}_6 = 7\ \text{Col}_2 + \text{Col}_3 + \text{Col}_5$$

(2 pts) (*a*) $\operatorname{rank}(R) =$ __4__

(4 pts) (*b*) Find R (there is only one possible answer). Clearly explain your reasoning to receive credit.

Solution. The column relations tell us that Col_4 and Col_6 are the nonpivot columns of R. It follows that $\text{Col}_1, \text{Col}_2, \text{Col}_3, \text{Col}_5$ are the pivot columns. Accounting for this taxonomy and the column relations gives

$$R = \begin{bmatrix} 1 & 0 & 0 & -5 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 7 \\ 0 & 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(9 pts) **Problem 12.** The augmented matrix below represents a system of linear equations in row echelon form whose variables are x_1, x_2, x_3, x_4, x_5.

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{matrix} \\ \left[\begin{array}{rrrrrr|r} -1 & 2 & 1 & -5 & 4 & -1 & 9 \\ 0 & 5 & 20 & 1 & -9 & 2 & 21 \\ 0 & 0 & 0 & 3 & -12 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 & -7 & -21 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Use the technique of *back-substitution* to find the general solution to this system. **You must use the method back-substitution to receive credit.**

Solution. The system is consistent because there is no pivot in the augmented column. The free variables are $x_3 = c_1$ and $x_5 = c_2$. The equations are then

$$\begin{array}{rcrcrcrcrcrcr} -x_1 & + & 2x_2 & + & c_1 & - & 5x_4 & + & 4c_2 & - & x_6 & = & 9 \\ & & 5x_2 & + & 20c_1 & + & x_4 & - & 9c_2 & + & 2x_6 & = & 21 \\ & & & & & & 3x_4 & - & 12c_2 & + & 2x_6 & = & 6 \\ & & & & & & & & & & -7x_6 & = & -21 \end{array}$$

The last equation tells us that $x_6 = 3$. We then solve for x_4 in the third equation as

$$x_4 = \frac{6 + 12c_2 - 2(3)}{3} = 4c_2$$

We then solve for x_2 in the second equation with

$$x_2 = \frac{21 - 20c_1 - (4c_2) + 9c_2 - 2(3)}{5} = \frac{15 - 20c_1 + 5c_2}{5} = 3 - 4c_1 + c_2$$

Finally, we solve for x_1 in the first equation

$$\begin{aligned} x_1 &= \frac{9 - 2(3 - 4c_1 + c_2) - c_1 + 5(4c_2) - 4c_2 + 3}{-1} \\ &= \frac{12 - 6 + 8c_1 - 2c_2 - c_1 + 20c_2 - 4c_2}{-1} \\ &= \frac{6 + 7c_1 + 14c_2}{-1} \\ &= -6 - 7c_1 - 14c_2 \end{aligned}$$

The general solution is then

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -7c_1 - 14c_2 - 6 \\ -4c_1 + c_2 + 3 \\ c_1 \\ 4c_2 \\ c_2 \\ 3 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \\ 0 \\ 0 \\ 0 \\ 3 \end{bmatrix} + c_1 \cdot \begin{bmatrix} -7 \\ -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \cdot \begin{bmatrix} -14 \\ 1 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix}$$

(9 pts) **Problem 13.** Consider the matrix A given by

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 2 & -1 \\ 5 & 15 & -20 & 10 & 0 & 5 \\ 1 & 3 & -2 & 4 & 2 & 1 \\ -3 & -9 & 11 & -7 & -1 & -3 \end{bmatrix}$$

Use the Gauß-Jordan algorithm to calculate rref(A).

You must label each row reduction properly and adhere to the steps of the algorithm to receive credit.

Solution. Following the algorithm, we have

$$\overset{A}{\begin{bmatrix} 0 & 0 & 0 & 1 & 2 & -1 \\ 5 & 15 & -20 & 10 & 0 & 5 \\ 1 & 3 & -2 & 4 & 2 & 1 \\ -3 & -9 & 11 & -7 & -1 & -3 \end{bmatrix}} \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} 5 & 15 & -20 & 10 & 0 & 5 \\ 0 & 0 & 0 & 1 & 2 & -1 \\ 1 & 3 & -2 & 4 & 2 & 1 \\ -3 & -9 & 11 & -7 & -1 & -3 \end{bmatrix}$$

$$\xrightarrow{(1/5)\cdot r_1 \to r_1} \begin{bmatrix} 1 & 3 & -4 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & -1 \\ 1 & 3 & -2 & 4 & 2 & 1 \\ -3 & -9 & 11 & -7 & -1 & -3 \end{bmatrix}$$

$$\xrightarrow{\substack{r_3 - r_1 \to r_3 \\ r_4 + 3\cdot r_1 \to r_4}} \begin{bmatrix} 1 & 3 & -4 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 2 & 2 & 2 & 0 \\ 0 & 0 & -1 & -1 & -1 & 0 \end{bmatrix}$$

$$\xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & 3 & -4 & 2 & 0 & 1 \\ 0 & 0 & 2 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & -1 & -1 & -1 & 0 \end{bmatrix}$$

$$\xrightarrow{(1/2)\cdot r_2 \to r_2} \begin{bmatrix} 1 & 3 & -4 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & -1 & -1 & -1 & 0 \end{bmatrix}$$

$$\xrightarrow{\substack{r_1 + 4\cdot r_2 \to r_1 \\ r_4 + r_2 \to r_4}} \begin{bmatrix} 1 & 3 & 0 & 6 & 4 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\substack{r_1 - 6\cdot r_3 \to r_1 \\ r_2 - r_3 \to r_2}} \overset{\text{rref}(A)}{\begin{bmatrix} 1 & 3 & 0 & 0 & -8 & 7 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}$$

Problem 14. Consider the following reduced row echelon form calculation.

$$\operatorname{rref}\left[\begin{array}{rrrrr|rrrrr} 5 & -2 & -1 & -2 & 1 & 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ -4 & 1 & 3 & 0 & 4 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -3 & 1 & 3 & 0 & 2 & 0 & 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{rrrrr|rrrrr} 1 & 0 & 0 & 0 & 0 & -4 & -10 & -3 & -2 & 3 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & * \\ 0 & 0 & 1 & 0 & 0 & -3 & -7 & -2 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & -11 & -25 & -6 & -4 & 5 \\ 0 & 0 & 0 & 0 & 1 & -2 & -5 & -1 & -1 & 1 \end{array}\right]$$

The notation indicates that A is the 5×5 matrix whose columns are the first five columns of the first augmented matrix in the equation. Note that the $(2, 10)$ entry of the reduced matrix is unknown and marked as $*$.

(4 pts) (*a*) Find the missing entry marked $*$. Clearly explain your reasoning to receive credit.

Solution. The data given is $\operatorname{rref}[A \mid I_5] = [I_5 \mid B]$. This means that $\operatorname{rref}(A) = I_5$, so A is nonsingular and $B = A^{-1}$. It follows that

$$\overset{A}{\left[\begin{array}{rrrrr} 5 & -2 & -1 & -2 & 1 \\ -2 & 1 & 0 & 1 & -1 \\ -4 & 1 & 3 & 0 & 4 \\ 1 & -1 & 2 & -1 & 0 \\ -3 & 1 & 3 & 0 & 2 \end{array}\right]} \overset{A^{-1}}{\left[\begin{array}{rrrrr} -4 & -10 & -3 & -2 & 3 \\ 1 & 1 & -1 & -1 & * \\ -3 & -7 & -2 & -1 & 2 \\ -11 & -25 & -6 & -4 & 5 \\ -2 & -5 & -1 & -1 & 1 \end{array}\right]} = \overset{I_5}{\left[\begin{array}{rrrrr} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]}$$

From here, there are lots of ways to recover $*$. For instance, we know that the inner product of the second column of A with the fifth column of A^{-1} is zero, which gives the equation

$$-2\,(3) + * + 5 - 1 = 0$$

It follows that $* = 6 - 5 + 1 = 2$.

(4 pts) (*b*) Find all solutions $\boldsymbol{x}$ to $A\boldsymbol{x} = \boldsymbol{b}$ for $\boldsymbol{b} = \begin{bmatrix} 0 & 0 & 1 & -1 & 0 \end{bmatrix}^\intercal$. Clearly explain your reasoning to receive credit.

Solution. There is exactly one solution to $A\boldsymbol{x} = \boldsymbol{b}$ because A is invertible. This solution is

$$\boldsymbol{x} = \overset{A^{-1}}{\left[\begin{array}{rrrrr} -4 & -10 & -3 & -2 & 3 \\ 1 & 1 & -1 & -1 & * \\ -3 & -7 & -2 & -1 & 2 \\ -11 & -25 & -6 & -4 & 5 \\ -2 & -5 & -1 & -1 & 1 \end{array}\right]} \overset{\boldsymbol{b}}{\left[\begin{array}{r} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{array}\right]} = \left[\begin{array}{r} -3 \\ -1 \\ -2 \\ -6 \\ -1 \end{array}\right] - \left[\begin{array}{r} -2 \\ -1 \\ -1 \\ -4 \\ -1 \end{array}\right] = \left[\begin{array}{r} -1 \\ 0 \\ -1 \\ -2 \\ 0 \end{array}\right]$$