Duke University

Math 218D-2

Matrices and Vectors

Exam I

Name: ______________________ *Unique ID:* ______________

I have adhered to the Duke Community Standard in completing this exam.

Signature: ______________________

May 30, 2024

- There are 100 points and 14 problems on this 100-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

(4 pts) **Problem 1.** Suppose $S = A + A^\intercal$ where A is 2024×2024. Show that S is symmetric. **Avoid circular logic to receive credit.**

Problem 2. Consider $A = \begin{bmatrix} * & * & 0 & 6 & 3 & 12 \\ * & * & 0 & 2 & 1 & 4 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & 0 & 6 & 3 & 12 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & 0 & 4 & 2 & 8 \\ * & * & 0 & 2 & 1 & 4 \end{bmatrix}$. Entries in the first two columns are unknown and marked as $*$.

(3 pts) (a) A has _______ rows and _______ columns. The notation $\mathbb{R}^a \xrightarrow{A} \mathbb{R}^b$ is valid for $a =$ _______ and $b =$ _______.

(2 pts) (b) The matrix-vector products $A\boldsymbol{v}$ and $A^\intercal\boldsymbol{w}$ only make sense if $\boldsymbol{v}$ has _______ coordinates and $\boldsymbol{w}$ has _______ coordinates.

(2 pts) (c) Only one of the following linear combinations of the columns of A produces the zero vector. Select this linear combination.

○ $2\ \mathrm{Col}_3 + \mathrm{Col}_4 - 2\ \mathrm{Col}_5$ ○ $\mathrm{Col}_4 + 2\ \mathrm{Col}_5$ ○ $\mathrm{Col}_5 - 3\ \mathrm{Col}_6$ ○ $\mathrm{Col}_3 - \mathrm{Col}_4$

(2 pts) (d) Each of the following options below describes a strategy for filling in the missing data in the first two columns of A. Select each strategy that would make A have "rank one" (each option is worth 0.5pts).

○ Set every $*$ equal to 0.

○ Set Col_1 equal to Col_4 and set Col_2 equal to Col_5.

○ Set every $*$ equal to 1.

○ Set every $*$ in Col_1 equal to 1 and every $*$ in Col_2 equal to 2.

(2 pts) (e) Suppose we set every $*$ equal to zero. Then which column is called the "first pivot column" of A?

○ Col_1 ○ Col_2 ○ Col_3 ○ Col_4 ○ Col_5

(2 pts) **Problem 3.** Suppose that A and B are matrices and that $\boldsymbol{v}$ is a vector with the following properties.

$$v \in \mathbb{R}^8 \qquad\qquad A\boldsymbol{v} \in \mathbb{R}^5 \qquad\qquad B(A\boldsymbol{v}) \in \mathbb{R}^{13}$$

Then A is ______ $\times$ ______ and B is ______ $\times$ ______.

Problem 4. Suppose that $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathcal{E}_A(\lambda)$ and let $\boldsymbol{v}$ be defined by the linear combination $\boldsymbol{v} = c_1 \cdot \boldsymbol{v}_1 + c_2 \cdot \boldsymbol{v}_2$.

(2 pts) (*a*) Select all of the following symbols from this problem that represent a *scalar quantity* (no partial credit here).

○ $\boldsymbol{v}_1$ ○ $\boldsymbol{v}_2$ ○ A ○ λ ○ $\boldsymbol{v}$ ○ c_1 ○ c_2

(2 pts) (*b*) The context of this problem requires which of the following adjectives to apply to A?

○ square ○ symmetric ○ rank one ○ identity ○ upper triangular

(4 pts) (*c*) Show that $\boldsymbol{v} \in \mathcal{E}_A(\lambda)$. **Avoid circular logic to receive credit.**

Problem 5. The equations below depict the results of multiplying a matrix A by three vectors $\boldsymbol{w}$, $\boldsymbol{x}$, and $\boldsymbol{y}$.

$$\left[\quad A \quad\right]\overset{\boldsymbol{w}}{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}} = \begin{bmatrix}5\\5\\5\\5\end{bmatrix} \qquad \left[\quad A \quad\right]\overset{\boldsymbol{x}}{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}} = \begin{bmatrix}21\\9\\11\\7\end{bmatrix} \qquad \left[\quad A \quad\right]\overset{\boldsymbol{y}}{\begin{bmatrix}0\\0\\0\\-1\end{bmatrix}} = \begin{bmatrix}25\\7\\12\\5\end{bmatrix}$$

(2 pts) (*a*) Which (if any) of the following vectors is an eigenvector of A? ○ $\boldsymbol{w}$ ○ $\boldsymbol{x}$ ○ $\boldsymbol{y}$ ○ none of these

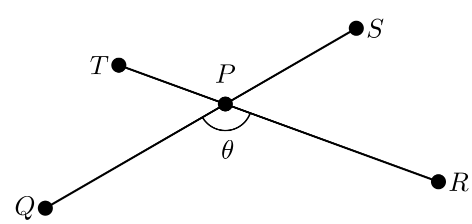
(4 pts) (*b*) Calculate $\mathrm{Col}_1 + \mathrm{Col}_3$, where Col_1 and Col_3 are the first and third columns of A. *Hint.* Start by explaining that this is the same thing as calculating $A\boldsymbol{v}$ for what vector $\boldsymbol{v}$?

(4 pts) **Problem 6.** Let G be the directed graph whose incidence matrix is given by

$$A = \begin{bmatrix} -1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Then $\chi(G) =$ _______, the number of connected components of G is _______, and the circuit rank of G is _______.

Problem 7. This figure depicts five points P, Q, R, S, and T in $\mathbb{R}^n$. The figure shows that the line segment connecting Q to S intersects the line segment connecting R to T at the point P. The angle θ depicted is the angle between the line segment connecting P to Q and the line segment connecting P to R.



(2 pts) (a) Only one of the following vectors is equal to $\overrightarrow{QR}$. Select this vector.

○ $\overrightarrow{QP} + \overrightarrow{PR}$ ○ $\overrightarrow{QP} - \overrightarrow{PR}$ ○ $\overrightarrow{QP} + \overrightarrow{PS}$ ○ $\overrightarrow{QP} + \overrightarrow{PQ}$

(2 pts) (b) Suppose that P and Q have coordinates $P(4, 5, 7, 10)$ and $Q(6, 6, 12, 11)$. Then $\|\overrightarrow{PQ}\| =$ _______.

(2 pts) (c) Only one of the following values is equal to $\|\overrightarrow{RT}\|$. Select this value.

○ $\|\overrightarrow{TP}\| + \|\overrightarrow{RP}\|$ ○ $\|\overrightarrow{RQ}\| + \|\overrightarrow{QT}\|$ ○ $\|\overrightarrow{RP}\| + \|\overrightarrow{PS}\| + \|\overrightarrow{ST}\|$ ○ $\|\overrightarrow{RP}\| - \|\overrightarrow{PT}\|$

(2 pts) (d) Suppose that $\langle \overrightarrow{PQ}, \overrightarrow{PR} \rangle = -5$. Then only one of the following statements about θ is correct. Select this statement.

○ θ is an acute angle ○ θ is an obtuse angle ○ θ is a right angle

○ we don't have enough information to conclude if θ is an acute, obtuse, or right angle

Problem 8. Suppose that A is a 2024×2023 matrix and that $\boldsymbol{v}$ and $\boldsymbol{w}$ are vectors in $\mathbb{R}^{2023}$ satisfying

$$\|\boldsymbol{v}\| = 5 \qquad \langle \boldsymbol{v}, \boldsymbol{w}\rangle = 5 \qquad \boldsymbol{v} \in \mathcal{E}_{A^\intercal A}(3) \qquad \boldsymbol{w} \in \mathcal{E}_{A^\intercal A}(0)$$

(3 pts) (a) Interpret the data in this problem by filling in the following blanks.

$$\langle \boldsymbol{v}, \boldsymbol{v}\rangle = \underline{\hspace{3cm}} \qquad A^\intercal A\boldsymbol{v} = \underline{\hspace{3cm}} \qquad A^\intercal A\boldsymbol{w} = \underline{\hspace{3cm}}$$

(5 pts) (b) Calculate $\|A(\boldsymbol{v} - \boldsymbol{w})\|^2$. Clearly show your steps to receive credit.

(5 pts) **Problem 9.** Consider a 2024×3 matrix A, so A looks like

$$A = \begin{bmatrix} | & | & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \\ | & | & | \end{bmatrix}$$

Suppose that each of the columns of A is a unit vector, that $\boldsymbol{a}_2$ is orthogonal to both $\boldsymbol{a}_1$ and $\boldsymbol{a}_3$, and that $\boldsymbol{a}_1$ forms an angle of $\theta = {}^{\pi}/{4}$ with $\boldsymbol{a}_3$. Find the Gramian $A^\intercal A$ of A.

Hint. Recall that $\cos({}^{\pi}/{4}) = {}^{\sqrt{2}}/{2}$.

Problem 10. Consider the calculation of the reduced row echelon form of the following 7×8 matrix A.

$$\operatorname{rref}\overbrace{\begin{bmatrix} -2 & -10 & -1 & 1 & -11 & 0 & -2 & -7 \\ -3 & -15 & -2 & 2 & -18 & 0 & -3 & -12 \\ 1 & 5 & 1 & 0 & 4 & -1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & -1 \\ -3 & -15 & -3 & 3 & -21 & 0 & -2 & -16 \\ -2 & -10 & -1 & 1 & -11 & 0 & -1 & -8 \\ 1 & 5 & 1 & -2 & 10 & 1 & 0 & 9 \end{bmatrix}}^{A} = \begin{bmatrix} 1 & 5 & 0 & 0 & 4 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -3 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(4 pts) (*a*) $\operatorname{rank}(A) =$ _______, $\operatorname{nullity}(A) =$ _______, $\operatorname{rank}(A^{\intercal}) =$ _______ and $\operatorname{nullity}(A^{\intercal}) =$ _______

(2 pts) (*b*) Which columns of A are the *nonpivot columns*? Select all that apply (no partial credit here).

○ Col_1 ○ Col_2 ○ Col_3 ○ Col_4 ○ Col_5 ○ Col_6 ○ Col_7 ○ Col_8

(2 pts) (*c*) Suppose $\boldsymbol{b} \in \mathbb{R}^7$ makes the system $A\boldsymbol{x} = \boldsymbol{b}$ consistent. Which of the variables in this system is *dependent*? Select all that apply (no partial credit here).

○ x_1 ○ x_2 ○ x_3 ○ x_4 ○ x_5 ○ x_6 ○ x_7 ○ x_8

(2 pts) (*d*) Which of the following adjectives correctly describes A? Select all that apply (no partial credit here).

○ full column rank ○ full row rank ○ full rank ○ rank deficient ○ nonsingular ○ singular

(2 pts) (*e*) The columns of A satisfy all of the following equations. However, according to our terminology from class, only one of the following equations is called a *column relation*. Select this equation.

○ $\text{Col}_5 = 4\ \text{Col}_1 - 3\ \text{Col}_4$ ○ $\text{Col}_5 = -\ \text{Col}_1 + \text{Col}_2 - 3\ \text{Col}_4$

○ $\text{Col}_1 = -3\ \text{Col}_2 + 12\ \text{Col}_4 + 4\ \text{Col}_5$ ○ $\text{Col}_2 = -3\,\text{Col}_1 + 6\,\text{Col}_4 + 2\ \text{Col}_5$

Problem 11. Let R be a 5×6 reduced row echelon form matrix with exactly two column relations given by

$$\text{Col}_4 = -5\ \text{Col}_1 + 2\ \text{Col}_3 \qquad\qquad \text{Col}_6 = 7\ \text{Col}_2 + \text{Col}_3 + \text{Col}_5$$

(2 pts) (*a*) $\operatorname{rank}(R) =$ _______

(4 pts) (*b*) Find R (there is only one possible answer). Clearly explain your reasoning to receive credit.

(9 pts) **Problem 12.** The augmented matrix below represents a system of linear equations in row echelon form whose variables are x_1, x_2, x_3, x_4, x_5.

$$\begin{array}{cccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \\ \end{array}$$

$$\left[\begin{array}{cccccc|c} -1 & 2 & 1 & -5 & 4 & -1 & 9 \\ 0 & 5 & 20 & 1 & -9 & 2 & 21 \\ 0 & 0 & 0 & 3 & -12 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 & -7 & -21 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Use the technique of *back-substitution* to find the general solution to this system. **You must use the method back-substitution to receive credit.**

(9 pts)**Problem 13.** Consider the matrix A given by

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 2 & -1 \\ 5 & 15 & -20 & 10 & 0 & 5 \\ 1 & 3 & -2 & 4 & 2 & 1 \\ -3 & -9 & 11 & -7 & -1 & -3 \end{bmatrix}$$

Use the Gauß-Jordan algorithm to calculate $\operatorname{rref}(A)$.

You must label each row reduction properly and adhere to the steps of the algorithm to receive credit.

Problem 14. Consider the following reduced row echelon form calculation.

$$\operatorname{rref}\overbrace{\left[\begin{array}{rrrrr|rrrrr} 5 & -2 & -1 & -2 & 1 & 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ -4 & 1 & 3 & 0 & 4 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -3 & 1 & 3 & 0 & 2 & 0 & 0 & 0 & 0 & 1 \end{array}\right]}^{A} = \left[\begin{array}{rrrrr|rrrrr} 1 & 0 & 0 & 0 & 0 & -4 & -10 & -3 & -2 & 3 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & * \\ 0 & 0 & 1 & 0 & 0 & -3 & -7 & -2 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & -11 & -25 & -6 & -4 & 5 \\ 0 & 0 & 0 & 0 & 1 & -2 & -5 & -1 & -1 & 1 \end{array}\right]$$

The notation indicates that A is the 5×5 matrix whose columns are the first five columns of the first augmented matrix in the equation. Note that the $(2, 10)$ entry of the reduced matrix is unknown and marked as $*$.

(4 pts) (a) Find the missing entry marked $*$. Clearly explain your reasoning to receive credit.

(4 pts) (b) Find all solutions $\boldsymbol{x}$ to $A\boldsymbol{x} = \boldsymbol{b}$ for $\boldsymbol{b} = \begin{bmatrix} 0 & 0 & 1 & -1 & 0 \end{bmatrix}^\intercal$. Clearly explain your reasoning to receive credit.