Duke University

Math 218D-2

Matrices and Vectors

Exam II

Name: Solutions

Unique ID:

I have adhered to the Duke Community Standard in completing this exam.

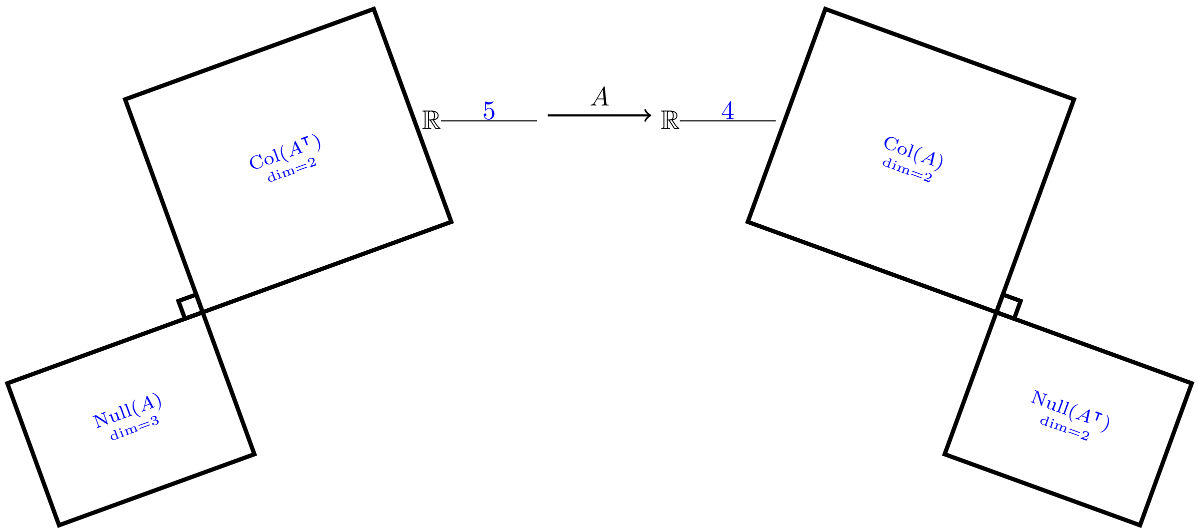
Signature:

June 20, 2024

- There are 100 points and 11 problems on this 100-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

Problem 1. Consider the factorization $\overset{E}{\begin{bmatrix} 0 & 0 & 1 & -5 \\ 1 & 0 & 0 & -3 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 3 \end{bmatrix}} \overset{A}{\begin{bmatrix} | & | & | & | & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 & \boldsymbol{a}_5 \\ | & | & | & | & | \end{bmatrix}} = \overset{R}{\begin{bmatrix} 1 & 5 & 0 & -3 & 1 \\ 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}$.

(5 pts) (*a*) Fill in every missing label in the picture of the four fundamental subspaces of A below, including the dimension of each fundamental subspace.



(2 pts) (*b*) The projection matrix P onto the left null space of A satisfies $\operatorname{trace}(P) =$ 2 .

(2 pts) (*c*) Which of the following statements correctly answers the question "Is $\lambda = 0$ an eigenvalue of E?"

○ No, because E is singular. ○ Yes, because E is singular. ○ Yes, because E is nonsingular.

√ No, because E is nonsingular. ○ No, because $\operatorname{trace}(E) \neq 0$.

(3 pts) (*d*) Find the *pivot basis* of Null(A).

Solution. The system $A\boldsymbol{x} = \boldsymbol{O}$ has five variables x_1, x_2, x_3, x_4, x_5. According to $R = \operatorname{rref}(A)$, the free variables are $x_2 = c_1$, $x_4 = c_2$, and $x_5 = c_3$. Solving for the dependent in terms of the free gives

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -5\,c_1 + 3\,c_2 - c_3 \\ c_1 \\ 2\,c_2 - 2\,c_3 \\ c_2 \\ c_3 \end{bmatrix} = c_1 \cdot \overset{\boldsymbol{x}_1}{\begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}} + c_2 \cdot \overset{\boldsymbol{x}_2}{\begin{bmatrix} 3 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}} + c_3 \cdot \overset{\boldsymbol{x}_3}{\begin{bmatrix} -1 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}}$$

The "pivot basis" of Null(A) is $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3\}$.

(2 pts) (*e*) Let R' be the matrix obtained by deleting all of the rows of zeros from R. Which (if any) of the following formulas for C satisfies the equation $A = CR'$?

○ $C = \begin{bmatrix} | & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 \\ | & | \end{bmatrix}$ ○ $C = \begin{bmatrix} | & | & | \\ \boldsymbol{a}_2 & \boldsymbol{a}_4 & \boldsymbol{a}_5 \\ | & | & | \end{bmatrix}$ ○ $C = \begin{bmatrix} | & | \\ \boldsymbol{a}_2 & \boldsymbol{a}_4 \\ | & | \end{bmatrix}$ √ $C = \begin{bmatrix} | & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_3 \\ | & | \end{bmatrix}$ ○ None of these.

(10 pts) **Problem 2.** Calculate $PA = LU$ for $A = \begin{bmatrix} -1 & -2 & -3 & 6 \\ 1 & 2 & 3 & -7 \\ 2 & 3 & 3 & -12 \\ -4 & -2 & 6 & 22 \end{bmatrix}$.

Solution. Following the algorithm from class, we have

$$\overset{A}{\begin{bmatrix} -1 & -2 & -3 & 6 \\ 1 & 2 & 3 & -7 \\ 2 & 3 & 3 & -12 \\ -4 & -2 & 6 & 22 \end{bmatrix}} \xrightarrow{\substack{\boldsymbol{r}_2 + \boldsymbol{r}_1 \to \boldsymbol{r}_2 \\ \boldsymbol{r}_3 + 2\cdot\boldsymbol{r}_1 \to \boldsymbol{r}_3 \\ \boldsymbol{r}_4 - 4\cdot\boldsymbol{r}_1 \to \boldsymbol{r}_4}} \overset{U}{\begin{bmatrix} -1 & -2 & -3 & 6 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & -3 & 0 \\ 0 & 6 & 18 & -2 \end{bmatrix}} \quad \overset{L}{\begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix}} \quad \overset{P}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

$$\xrightarrow{\boldsymbol{r}_2 \leftrightarrow \boldsymbol{r}_3} \begin{bmatrix} -1 & -2 & -3 & 6 \\ 0 & -1 & -3 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 6 & 18 & -2 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{\boldsymbol{r}_4 + 6\cdot\boldsymbol{r}_2 \to \boldsymbol{r}_4} \begin{bmatrix} -1 & -2 & -3 & 6 \\ 0 & -1 & -3 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -2 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 4 & -6 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{\boldsymbol{r}_4 - 2\cdot\boldsymbol{r}_3 \to \boldsymbol{r}_4} \begin{bmatrix} -1 & -2 & -3 & 6 \\ 0 & -1 & -3 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 4 & -6 & 2 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This gives our desired factorization

$$\overset{P}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \overset{A}{\begin{bmatrix} -1 & -2 & -3 & 6 \\ 1 & 2 & 3 & -7 \\ 2 & 3 & 3 & -12 \\ -4 & -2 & 6 & 22 \end{bmatrix}} = \overset{L}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 4 & -6 & 2 & 1 \end{bmatrix}} \overset{U}{\begin{bmatrix} -1 & -2 & -3 & 6 \\ 0 & -1 & -3 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

(6 pts) **Problem 3.** Find a matrix B satisfying $\text{Null}(B) = \text{Col}(A)$ where $A = \begin{bmatrix} 1 & 0 & 1 & -3 & -5 \\ 1 & 1 & 2 & -2 & -3 \\ 2 & 1 & 3 & -4 & -6 \\ -1 & -1 & -2 & 3 & 5 \end{bmatrix}$.

Solution. The criteria for $\boldsymbol{b} = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \end{bmatrix}^\intercal$ to be in $\text{Col}(A)$ is that the augmented matrix $[A \mid \boldsymbol{b}]$ is a consistent system. The consistency of this system is resolved with row reductions

$$\left[\begin{array}{rrrrr|r} 1 & 0 & 1 & -3 & -5 & b_1 \\ 1 & 1 & 2 & -2 & -3 & b_2 \\ 2 & 1 & 3 & -4 & -6 & b_3 \\ -1 & -1 & -2 & 3 & 5 & b_4 \end{array}\right] \xrightarrow[\substack{\boldsymbol{r}_4 + \boldsymbol{r}_1 \to \boldsymbol{r}_4}]{\substack{\boldsymbol{r}_2 - \boldsymbol{r}_1 \to \boldsymbol{r}_2 \\ \boldsymbol{r}_3 - 2\cdot\boldsymbol{r}_1 \to \boldsymbol{r}_3}} \left[\begin{array}{rrrrr|r} 1 & 0 & 1 & -3 & -5 & b_1 \\ 0 & 1 & 1 & 1 & 2 & -b_1 + b_2 \\ 0 & 1 & 1 & 2 & 4 & -2\,b_1 + b_3 \\ 0 & -1 & -1 & 0 & 0 & b_1 + b_4 \end{array}\right]$$

$$\xrightarrow{\substack{\boldsymbol{r}_3 - \boldsymbol{r}_2 \to \boldsymbol{r}_3 \\ \boldsymbol{r}_4 + \boldsymbol{r}_2 \to \boldsymbol{r}_4}} \left[\begin{array}{rrrrr|r} 1 & 0 & 1 & -3 & -5 & b_1 \\ 0 & 1 & 1 & 1 & 2 & -b_1 + b_2 \\ 0 & 0 & 0 & 1 & 2 & -b_1 - b_2 + b_3 \\ 0 & 0 & 0 & 1 & 2 & b_2 + b_4 \end{array}\right]$$

$$\xrightarrow{\boldsymbol{r}_4 - \boldsymbol{r}_3 \to \boldsymbol{r}_4} \left[\begin{array}{rrrrr|r} 1 & 0 & 1 & -3 & -5 & b_1 \\ 0 & 1 & 1 & 1 & 2 & -b_1 + b_2 \\ 0 & 0 & 0 & 1 & 2 & -b_1 - b_2 + b_3 \\ 0 & 0 & 0 & 0 & 0 & b_1 + 2\,b_2 - b_3 + b_4 \end{array}\right]$$

Here, we find that whether or not $[A \mid \boldsymbol{b}]$ is consistent is identical to the question of whether or not $b_1 + 2\,b_2 - b_3 + b_4 = 0$. This means that $\text{Col}(A) = \text{Null}(B)$ where $B = \begin{bmatrix} 1 & 2 & -1 & 1 \end{bmatrix}$.

Problem 4. Suppose $B = \begin{bmatrix} | & | & | \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \\ | & | & | \end{bmatrix}$ is $n \times 3$ and A is $m \times n$ such that $\{A\boldsymbol{v}_1, A\boldsymbol{v}_2, A\boldsymbol{v}_3\}$ is independent.

(5 pts) (a) Show that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is independent.

Solution. Suppose that $c_1 \cdot \boldsymbol{v}_1 + c_2 \cdot \boldsymbol{v}_2 + c_3 \cdot \boldsymbol{v}_3 = \boldsymbol{O}$. Multiplying by A gives

$$c_1 \cdot A\boldsymbol{v}_1 + c_2 \cdot A\boldsymbol{v}_2 + c_3 \cdot A\boldsymbol{v}_3 = \boldsymbol{O}$$

We are told that $\{A\boldsymbol{v}_1, A\boldsymbol{v}_2, A\boldsymbol{v}_3\}$ is independent, so we must have $c_1 = c_2 = c_3 = 0$.

(5 pts) (b) The data from this problem along with the result from part (a) allows us to infer which of the following statements? Select all that apply (one point each).

√ B has full column rank √ $\text{rank}(AB) = 3$ ◯ B has full row rank

◯ AB has full row rank √ $\text{nullity}(B) = 0$

(10 pts) **Problem 5.** Use the Gram-Schmidt algorithm to calculate Q in $A = QR$ where $A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 2 & 1 \\ 0 & 0 & -3 \\ -2 & -2 & -4 \\ -2 & -6 & 0 \end{bmatrix}$.

Solution. Call the columns $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ and start with

$$\boldsymbol{w}_1 = \boldsymbol{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \\ -2 \end{bmatrix}$$

The formula for $\boldsymbol{w}_2$ is $\boldsymbol{w}_2 = \boldsymbol{v}_2 - \operatorname{proj}_{\boldsymbol{w}_1}(\boldsymbol{v}_2)$, which is

$$\boldsymbol{w}_2 = \boldsymbol{v}_2 - \frac{\langle \boldsymbol{v}_2, \boldsymbol{w}_1 \rangle}{\langle \boldsymbol{w}_1, \boldsymbol{w}_1 \rangle} \boldsymbol{w}_1 = \boldsymbol{v}_2 - \frac{18}{9} \boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -2 \\ -6 \end{bmatrix} - 2 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \\ -2 \end{bmatrix}$$

The formula for $\boldsymbol{w}_3$ is $\boldsymbol{w}_3 = \boldsymbol{v}_3 - \operatorname{proj}_{w_1}(\boldsymbol{v}_3) - \operatorname{proj}_{w_2}(\boldsymbol{v}_3)$, which is

$$\boldsymbol{w}_3 = \boldsymbol{v}_3 - \frac{\langle \boldsymbol{v}_3, \boldsymbol{w}_1 \rangle}{\langle \boldsymbol{w}_1, \boldsymbol{w}_1 \rangle} \boldsymbol{w}_1 - \frac{\langle \boldsymbol{v}_3, \boldsymbol{w}_2 \rangle}{\langle \boldsymbol{w}_2, \boldsymbol{w}_2 \rangle} \boldsymbol{w}_2 = \boldsymbol{v}_3 - \frac{9}{9} \boldsymbol{w}_1 - \frac{-9}{9} \boldsymbol{w}_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \\ -4 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \\ -2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -3 \\ 0 \\ 0 \end{bmatrix}$$

The columns of Q are the normalizations of these vectors.

$$\boldsymbol{q}_1 = \frac{1}{\|\boldsymbol{w}_1\|} \boldsymbol{w}_1 = \frac{1}{3} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \\ -2 \end{bmatrix} \qquad \boldsymbol{q}_2 = \frac{1}{\|\boldsymbol{w}_2\|} \boldsymbol{w}_2 = \frac{1}{3} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \\ -2 \end{bmatrix} \qquad \boldsymbol{q}_3 = \frac{1}{\|\boldsymbol{w}_3\|} \boldsymbol{w}_3 = \frac{1}{3} \begin{bmatrix} 0 \\ 0 \\ -3 \\ 0 \\ 0 \end{bmatrix}$$

This gives

$$Q = \frac{1}{3} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -3 \\ -2 & 2 & 0 \\ -2 & -2 & 0 \end{bmatrix}$$

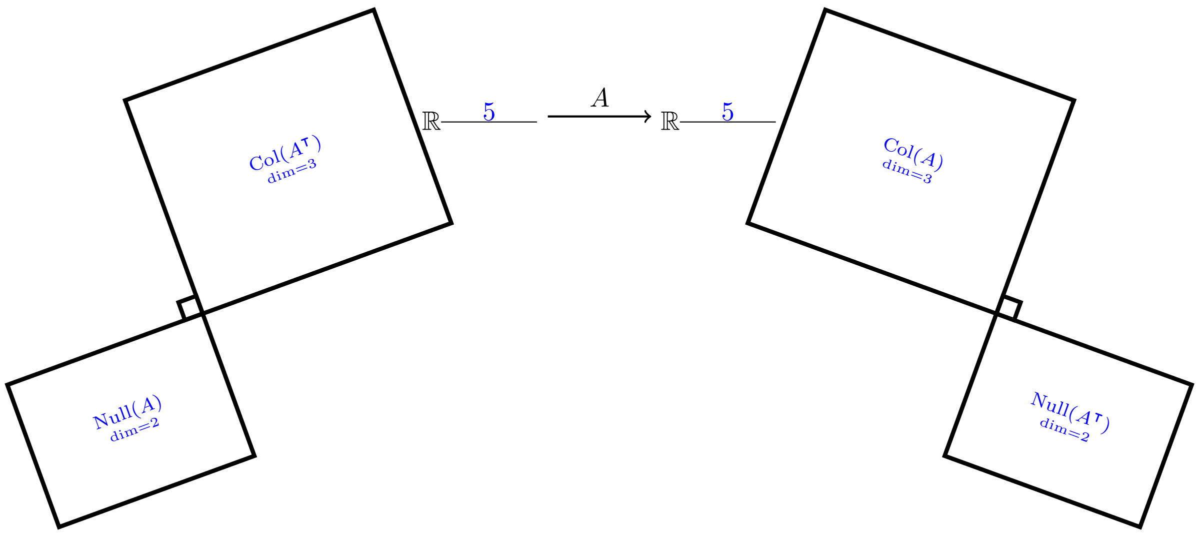
Problem 6. The equation below depicts $A = QR$ where A is the incidence matrix of a directed graph G along with a vector $\boldsymbol{b}$.

$$\left[\begin{array}{c} \\ \quad\quad A \quad\quad \\ \\ \end{array}\right] = \overset{Q}{\begin{bmatrix} -\sqrt{2}/2 & 0 & 0 \\ \sqrt{2}/2 & 0 & 0 \\ 0 & -\sqrt{2}/2 & -\sqrt{6}/6 \\ 0 & \sqrt{2}/2 & -\sqrt{6}/6 \\ 0 & 0 & \sqrt{6}/3 \end{bmatrix}} \overset{R}{\begin{bmatrix} \sqrt{2} & -\sqrt{2} & * & * & 0 \\ 0 & 0 & \sqrt{2} & * & * \\ 0 & 0 & 0 & \sqrt{6}/2 & * \end{bmatrix}} \qquad \boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

Note that the matrix R is missing several entries marked as $*$.

(6 pts) (*a*) Calculate $h_0(G)$ and $h_1(G)$. Show your work in the space provided and fill in your answers in the blanks below.

Solution. The given equation is $A = QR$ where Q is 5×3 and R is 3×5. This tells us that A is 5×5 with rank three. The picture of the four fundamental subspaces is



From class we know that $h_0(G) = \dim \operatorname{Null}(A^\intercal)$ and $h_1(G) = \dim \operatorname{Null}(A)$.

$h_0(G) =$ ___2___ $h_1(G) =$ ___2___

(6 pts) (*b*) Find the projection of $\boldsymbol{b}$ onto $\operatorname{Col}(A)$. Use this projection to decide if $A\boldsymbol{x} = \boldsymbol{b}$ is consistent.

Solution. From class we know that the projection matrix onto $\operatorname{Col}(A)$ is $P = QQ^\intercal$. Our desired projection is then

$$P\boldsymbol{b} = \left[\begin{array}{c} \\ \quad Q \quad \\ \\ \end{array}\right] \overset{Q^\intercal}{\begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & -\sqrt{6}/6 & -\sqrt{6}/6 & \sqrt{6}/3 \end{bmatrix}} \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \overset{Q}{\begin{bmatrix} -\sqrt{2}/2 & 0 & 0 \\ \sqrt{2}/2 & 0 & 0 \\ 0 & -\sqrt{2}/2 & -\sqrt{6}/6 \\ 0 & \sqrt{2}/2 & -\sqrt{6}/6 \\ 0 & 0 & \sqrt{6}/3 \end{bmatrix}} \begin{bmatrix} 0 \\ -\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

Our equation shows that $P\boldsymbol{b} \neq \boldsymbol{b}$, so $A\boldsymbol{x} = \boldsymbol{b}$ is *inconsistent*.

(7 pts) **Problem 7.** Consider the matrix $Q = Q_1Q_2$ where Q_1 is $m \times n$ with orthonormal columns and Q_2 is $n \times \ell$ with orthonormal columns. Show that Q has orthonormal columns.

Solution. We are told that Q_1 and Q_2 have orthonormal columns, which means that $Q_1^\intercal Q_1 = I_n$ and $Q_2^\intercal Q_2 = I_\ell$. We wish to show that $Q = Q_1Q_2$ has orthonormal columns. To do so, note that

$$Q^\intercal Q = (Q_1Q_2)^\intercal (Q_1Q_2) = Q_2^\intercal Q_1^\intercal Q_1 Q_2 = Q_2^\intercal I_n Q_2 = Q_2^\intercal Q_2 = I_\ell$$

(7 pts) **Problem 8.** Let P be the projection matrix onto $V = \operatorname{Span}\begin{bmatrix}1 & -1 & -1 & 1\end{bmatrix}^\intercal, \begin{bmatrix}1 & 0 & -1 & 1\end{bmatrix}^\intercal$. The projection formula expresses P as the product of three matrices as follows.

$$\begin{bmatrix} & & & \\ & & & \\ & P & & \\ & & & \\ & & & \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 4/3 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 0 & -1 & 1 \end{bmatrix}$$

Fill in the missing entries of these three matrices.

Solution. The projection formula is $P = X(X^\intercal X)^{-1}X^\intercal$ where the columns of X form a basis of V. The vector space V is defined as the span of two nonparallel vectors, so these two vectors form a basis. We need only calculate $(X^\intercal X)^{-1}$.

$$\overset{X^\intercal}{\begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 0 & -1 & 1 \end{bmatrix}} \overset{X}{\begin{bmatrix} 1 & 1 \\ -1 & 0 \\ -1 & -1 \\ 1 & 1 \end{bmatrix}} = \begin{bmatrix} 4 & 3 \\ 3 & 3 \end{bmatrix} \qquad (X^\intercal X)^{-1} = \frac{1}{3}\begin{bmatrix} 3 & -3 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 4/3 \end{bmatrix}$$

Here, we have used the adjugate formula for 2×2 inverses.

(7 pts) **Problem 9.** Suppose that A is a 3×3 complex matrix satisfying $A^* = i\,A$ and consider the vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{C}^3$ satisfying

$$\boldsymbol{v} = \begin{bmatrix} -4-2\,i \\ 4-2\,i \\ -1-i \end{bmatrix} \qquad \boldsymbol{w} = \begin{bmatrix} 1-i \\ i \\ 1+i \end{bmatrix} \qquad A\boldsymbol{v} = \begin{bmatrix} 2 \\ 2\,i \\ 2 \end{bmatrix}$$

Find $\langle \boldsymbol{v}, A\boldsymbol{w} \rangle$.

Solution. Here, we have

$$\begin{aligned}
\langle \boldsymbol{v}, A\boldsymbol{w} \rangle &= \langle A^*\boldsymbol{v}, \boldsymbol{w} \rangle \\
&= \langle i\,A\boldsymbol{v}, \boldsymbol{w} \rangle \\
&= \bar{i} \cdot \langle A\boldsymbol{v}, \boldsymbol{w} \rangle \\
&= -i \cdot \langle \begin{bmatrix} 2 & 2i & 2 \end{bmatrix}^\intercal, \begin{bmatrix} 1-i & i & 1+i \end{bmatrix}^\intercal \rangle \\
&= -i \cdot \{\bar{2} \cdot (1-i) + \overline{2\,i} \cdot i + \bar{2} \cdot (1+i)\} \\
&= -i \cdot \{2 \cdot (1-i) - 2\,i \cdot i + 2 \cdot (1+i)\} \\
&= -i \cdot \{2 - 2\,i - 2\,i^2 + 2 + 2\,i\} \\
&= -i \cdot \{2 - 2\,i + 2 + 2 + 2\,i\} \\
&= -i \cdot \{6\} \\
&= -6\,i
\end{aligned}$$

(7 pts) **Problem 10.** The data below depicts a system of linear equations along with three determinant calculations.

$$\begin{array}{rcrcrcr} 3\,x & + & 5\,y & - & 11\,z & = & -4 \\ x & + & y & - & 4\,z & = & 0 \\ x & - & y & - & 3\,z & = & -8 \end{array} \qquad \begin{vmatrix} 3 & 5 & -11 \\ 1 & 1 & -4 \\ 1 & -1 & -3 \end{vmatrix} = -4 \qquad \begin{vmatrix} -4 & 5 & -11 \\ 0 & 1 & -4 \\ -8 & -1 & -3 \end{vmatrix} = 100 \qquad \begin{vmatrix} 3 & 5 & -4 \\ 1 & 1 & 0 \\ 1 & -1 & -8 \end{vmatrix} = 24$$

Fill in the blanks below with the values of x, y, and z that solve this system (each of these values will simplify to an integer quantity).

$x =$ <u>-25</u> $\qquad y =$ <u>1</u> $\qquad z =$ <u>-6</u>

Use the space below for any necessary scratch work.

Solution. In the notation for Cramer's rule, we are given the system $A\boldsymbol{x} = \boldsymbol{b}$ along with $\det(A) = -4$, $\det(A_1) = 100$, and $\det(A_3) = 24$. Immediately we find $x = \frac{\det(A_1)}{\det(A)} = \frac{100}{-4} = -25$ and $z = \frac{\det(A_3)}{\det(A)} = \frac{24}{-4} = -6$. We'll need to then calculate $\det(A_2)$ to find y.

$$\overset{A_2}{\begin{vmatrix} 3 & -4 & -11 \\ 1 & 0 & -4 \\ 1 & -8 & -3 \end{vmatrix}} \xlongequal{\boldsymbol{r}_3 - 2\cdot\boldsymbol{r}_1 \to \boldsymbol{r}_3} \begin{vmatrix} 3 & -4 & -11 \\ 1 & 0 & -4 \\ -5 & 0 & 19 \end{vmatrix} \xlongequal{\text{Col}_3} -(-4)\begin{vmatrix} 1 & -4 \\ -5 & 19 \end{vmatrix} = 4 \cdot (19 - 20) = -4$$

It follows that $y = \frac{\det(A_2)}{\det(A)} = \frac{-4}{-4} = 1$.

(10 pts)**Problem 11.** Find bases of all eigenspaces of $A = \begin{bmatrix} 6 & 6 & -4 \\ -3 & -3 & 5 \\ 0 & 0 & 3 \end{bmatrix}$.

Solution. We must start by finding the *eigenvalues* of A, which requires us to factor the characteristic polynomial.

$$\chi_A(t) = \begin{vmatrix} t-6 & -6 & 4 \\ 3 & t+3 & -5 \\ 0 & 0 & t-3 \end{vmatrix} \overset{\text{Row}_3}{=\!=\!=} (t-3)\begin{vmatrix} t-6 & -6 \\ 3 & t+3 \end{vmatrix} = (t-3)\{(t+3)(t-6)+18\} = (t-3)\{t^2-3t\} = (t-3)^2 t$$

This demonstrates that E-Vals$(A) = \{0, 3\}$. For $\lambda = 0$, the eigenspace is

$$\mathcal{E}_A(0) = \operatorname{Null} \overbrace{\begin{bmatrix} -6 & -6 & 4 \\ 3 & 3 & -5 \\ 0 & 0 & -3 \end{bmatrix}}^{0 \cdot I_3 - A} = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$$

For $\lambda = 3$, the eigenspace is

$$\mathcal{E}_A(3) = \operatorname{Null} \overbrace{\begin{bmatrix} -3 & -6 & 4 \\ 3 & 6 & -5 \\ 0 & 0 & 0 \end{bmatrix}}^{3 \cdot I_3 - A} = \operatorname{Span}\left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \right\}$$