Duke University

Math 218D-2

Matrices and Vectors

Exam I

Name: Solutions

Unique ID:

I have adhered to the Duke Community Standard in completing this exam.

Signature:

July 18, 2024

- There are 100 points and 10 problems on this 100-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

Duke UNIVERSITY MATH

Problem 1. Suppose that A is a matrix satisfying the matrix-vector product $\begin{bmatrix} & A & \end{bmatrix}\overset{\boldsymbol{v}}{\begin{bmatrix}-1\\-1\\0\\1\end{bmatrix}} = \begin{bmatrix}1\\-2\\-1\\0\\-1\end{bmatrix}$.

(2 pts) (a) The matrix A has 5 rows and 4 columns.

(3 pts) (b) Which of the following vectors is *orthogonal* to the vector $\boldsymbol{v}$?

○ 1st row of A ○ 2nd row of A ○ 3rd row of A √ 4th row of A ○ 3rd column of A

(3 pts) (c) Which of the following vectors forms an *acute* angle with $\boldsymbol{v}$?

√ 1st row of A ○ 2nd row of A ○ 3rd row of A ○ 4th row of A ○ 3rd column of A

(4 pts) (d) Fill in the entries of this matrix product: $\begin{bmatrix} & A & \end{bmatrix}\begin{bmatrix}1 & 2 & 0 & 2\\1 & 2 & 0 & 2\\0 & 0 & 0 & 0\\-1 & -2 & 0 & -2\end{bmatrix} = \begin{bmatrix}-1 & -2 & 0 & -2\\2 & 4 & 0 & 4\\1 & 2 & 0 & 2\\0 & 0 & 0 & 0\\1 & 2 & 0 & 2\end{bmatrix}$

(4 pts) (e) Calculate $\langle A^\intercal \boldsymbol{w}, \boldsymbol{v}\rangle$ where $\boldsymbol{w} = \begin{bmatrix}1 & 1 & 1 & 1 & 1\end{bmatrix}^\intercal$. Clearly explain your reasoning to receive credit.

Solution. Here we use the adjoint formula

$$\langle A^\intercal \boldsymbol{w}, \boldsymbol{v}\rangle = \langle \boldsymbol{w}, A\boldsymbol{v}\rangle = \langle \begin{bmatrix}1 & 1 & 1 & 1 & 1\end{bmatrix}^\intercal, \begin{bmatrix}1 & -2 & -1 & 0 & -1\end{bmatrix}^\intercal\rangle = -3$$

(6 pts) **Problem 2.** Let $S = A^\intercal A + BCB$ where A is a 5×2024 matrix and B and C are 2024×2024 symmetric matrices. Show that S is symmetric.

Solution. We are given that B and C are symmetric, which means $B^\intercal = B$ and $C^\intercal = C$. To demonstrate that S is symmetric, we must show that $S^\intercal = S$. To do so, note that

$$S^\intercal = (A^\intercal A + BCB)^\intercal = (A^\intercal A)^\intercal + (BCB)^\intercal = A^\intercal (A^\intercal)^\intercal B^\intercal C^\intercal B^\intercal = A^\intercal A + BCB = S$$

(5 pts) **Problem 3.** Suppose that $\boldsymbol{v}$, $\boldsymbol{w}_1$, and $\boldsymbol{w}_2$ are vectors in $\mathbb{R}^n$ such that $\boldsymbol{v}$ is orthogonal to both $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$. Let $\boldsymbol{w} = c_1 \cdot \boldsymbol{w}_1 + c_2 \cdot \boldsymbol{w}_2$. Show that $\boldsymbol{v}$ is orthogonal to $\boldsymbol{w}$.

You must avoid circular reasoning to receive credit.

Solution. We are told that $\boldsymbol{v}$ is orthogonal to $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$, so

$$\langle \boldsymbol{v}, \boldsymbol{w}_1 \rangle = 0 \qquad\qquad \langle \boldsymbol{v}, \boldsymbol{w}_2 \rangle = 0$$

We wish to demonstrate that $\boldsymbol{v}$ is orthogonal to $\boldsymbol{w}$, which requires us to validate the equation $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0$. To do so, note that

$$\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \langle \boldsymbol{v}, c_1 \cdot \boldsymbol{w}_1 + c_2 \cdot \boldsymbol{w}_2 \rangle = c_1 \cdot \langle \boldsymbol{v}, \boldsymbol{w}_1 \rangle + c_2 \cdot \langle \boldsymbol{v}, \boldsymbol{w}_2 \rangle = c_1 \cdot 0 + c_2 \cdot 0 = 0$$

(5 pts) **Problem 4.** Suppose that A is $n \times n$ and that $\boldsymbol{v} \in \mathcal{E}_A(\lambda)$ and let $X_\lambda = \lambda \cdot I_n - A$. Calculate $X_\lambda \boldsymbol{v}$. Clearly explain your reasoning to receive credit.

Solution. We are told that $\boldsymbol{v} \in \mathcal{E}_A(\lambda)$, which means $A\boldsymbol{v} = \lambda \cdot \boldsymbol{v}$. It follows that

$$X_\lambda \boldsymbol{v} = (\lambda \cdot I_n - A)\boldsymbol{v} = \lambda \cdot \boldsymbol{v} - A\boldsymbol{v} = \lambda \cdot \boldsymbol{v} - \lambda \cdot \boldsymbol{v} = \boldsymbol{O}$$

(10 pts) **Problem 5.** Consider the matrix A given by

$$A = \begin{bmatrix} 2 & -2 & 2 & 4 & 4 & 6 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 1 & -1 & 1 & 2 & 3 & 4 \\ 0 & -2 & 2 & 2 & 0 & 2 \\ 0 & 1 & -1 & -1 & 0 & -1 \end{bmatrix}$$

Use the Gauß-Jordan algorithm to calculate $\operatorname{rref}(A)$.

You must label each row reduction properly and adhere to the steps of the algorithm to receive credit.

Solution. Following the algorithm, we have

$$\overbrace{\begin{bmatrix} 2 & -2 & 2 & 4 & 4 & 6 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 1 & -1 & 1 & 2 & 3 & 4 \\ 0 & -2 & 2 & 2 & 0 & 2 \\ 0 & 1 & -1 & -1 & 0 & -1 \end{bmatrix}}^{A} \xrightarrow{(1/2)\cdot \boldsymbol{r}_1 \to \boldsymbol{r}_1} \begin{bmatrix} 1 & -1 & 1 & 2 & 2 & 3 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 1 & -1 & 1 & 2 & 3 & 4 \\ 0 & -2 & 2 & 2 & 0 & 2 \\ 0 & 1 & -1 & -1 & 0 & -1 \end{bmatrix}$$

$$\xrightarrow{\boldsymbol{r}_3 - \boldsymbol{r}_1 \to \boldsymbol{r}_3} \begin{bmatrix} 1 & -1 & 1 & 2 & 2 & 3 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & -2 & 2 & 2 & 0 & 2 \\ 0 & 1 & -1 & -1 & 0 & -1 \end{bmatrix}$$

$$\xrightarrow{\boldsymbol{r}_2 \leftrightarrow \boldsymbol{r}_4} \begin{bmatrix} 1 & -1 & 1 & 2 & 2 & 3 \\ 0 & -2 & 2 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & -1 & -1 & 0 & -1 \end{bmatrix}$$

$$\xrightarrow{(-1/2)\cdot \boldsymbol{r}_2 \to \boldsymbol{r}_2} \begin{bmatrix} 1 & -1 & 1 & 2 & 2 & 3 \\ 0 & 1 & -1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & -1 & -1 & 0 & -1 \end{bmatrix}$$

$$\xrightarrow[\boldsymbol{r}_5 - \boldsymbol{r}_2 \to \boldsymbol{r}_5]{\boldsymbol{r}_1 + \boldsymbol{r}_2 \to \boldsymbol{r}_1} \begin{bmatrix} 1 & 0 & 0 & 1 & 2 & 2 \\ 0 & 1 & -1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[\boldsymbol{r}_4 + \boldsymbol{r}_3 \to \boldsymbol{r}_4]{\boldsymbol{r}_1 - 2\cdot \boldsymbol{r}_3 \to \boldsymbol{r}_1} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(10 pts) **Problem 6.** Calculate $PA = LU$ for $A = \begin{bmatrix} 2 & 5 & 3 & 1 & 3 \\ 0 & 0 & 0 & 3 & 5 \\ -2 & -2 & 2 & -4 & 2 \\ 4 & 7 & 1 & 2 & 7 \end{bmatrix}$.

Solution. Following the algorithm from class, we have

$$\overset{A}{\begin{bmatrix} 2 & 5 & 3 & 1 & 3 \\ 0 & 0 & 0 & 3 & 5 \\ -2 & -2 & 2 & -4 & 2 \\ 4 & 7 & 1 & 2 & 7 \end{bmatrix}} \xrightarrow[\boldsymbol{r}_4 - 2\cdot\boldsymbol{r}_1 \to \boldsymbol{r}_4]{\boldsymbol{r}_3 + \boldsymbol{r}_1 \to \boldsymbol{r}_3} \overset{U}{\begin{bmatrix} 2 & 5 & 3 & 1 & 3 \\ 0 & 0 & 0 & 3 & 5 \\ 0 & 3 & 5 & -3 & 5 \\ 0 & -3 & -5 & 0 & 1 \end{bmatrix}} \quad \overset{L}{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}} \quad \overset{P}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

$$\xrightarrow{\boldsymbol{r}_2 \leftrightarrow \boldsymbol{r}_3} \begin{bmatrix} 2 & 5 & 3 & 1 & 3 \\ 0 & 3 & 5 & -3 & 5 \\ 0 & 0 & 0 & 3 & 5 \\ 0 & -3 & -5 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{\boldsymbol{r}_4 + \boldsymbol{r}_2 \to \boldsymbol{r}_4} \begin{bmatrix} 2 & 5 & 3 & 1 & 3 \\ 0 & 3 & 5 & -3 & 5 \\ 0 & 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & -3 & 6 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{\boldsymbol{r}_4 + \boldsymbol{r}_3 \to \boldsymbol{r}_4} \begin{bmatrix} 2 & 5 & 3 & 1 & 3 \\ 0 & 3 & 5 & -3 & 5 \\ 0 & 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 0 & 11 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & -1 & -1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This gives our desired factorization

$$\overset{P}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \overset{A}{\begin{bmatrix} 2 & 5 & 3 & 1 & 3 \\ 0 & 0 & 0 & 3 & 5 \\ -2 & -2 & 2 & -4 & 2 \\ 4 & 7 & 1 & 2 & 7 \end{bmatrix}} = \overset{L}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & -1 & -1 & 1 \end{bmatrix}} \overset{U}{\begin{bmatrix} 2 & 5 & 3 & 1 & 3 \\ 0 & 3 & 5 & -3 & 5 \\ 0 & 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 0 & 11 \end{bmatrix}}$$

Problem 7. Consider the matrices A, B, and Y given by

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 2 & 1 & 3 \\ -2 & -2 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & -1 & 0 & 1 \\ -2 & 1 & 0 & -1 \\ -2 & 1 & 1 & 0 \end{bmatrix} \qquad Y = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(5 pts) (a) Calculate AB. The matrix AB is __4__ $\times$ __4__ and trace(AB) = __3__.

Solution.
$$\overset{A}{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 2 & 1 & 3 \\ -2 & -2 & -2 \end{bmatrix}} \overset{B}{\begin{bmatrix} 3 & -1 & 0 & 1 \\ -2 & 1 & 0 & -1 \\ -2 & 1 & 1 & 0 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & -2 & -1 \\ -2 & 2 & 3 & 1 \\ 2 & -2 & -2 & 0 \end{bmatrix}$$

(5 pts) (b) Calculate BA. The matrix BA is __3__ $\times$ __3__ and trace(BA) = __3__.

Solution.
$$\overset{B}{\begin{bmatrix} 3 & -1 & 0 & 1 \\ -2 & 1 & 0 & -1 \\ -2 & 1 & 1 & 0 \end{bmatrix}} \overset{A}{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 2 & 1 & 3 \\ -2 & -2 & -2 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(5 pts) (c) If possible, find a matrix X satisfying $XA = Y$. If this is not possible, then explain why.

Solution. Our work above tells us that

$$BA = I_3$$

Since Y is __5__ $\times$ __3__, we can multiply this equation on the left by Y to obtain

$$YBA = YI_3 = Y$$

So, the matrix

$$X = \overset{Y}{\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}} \overset{B}{\begin{bmatrix} 3 & -1 & 0 & 1 \\ -2 & 1 & 0 & -1 \\ -2 & 1 & 1 & 0 \end{bmatrix}} = \begin{bmatrix} 5 & -2 & 0 & 2 \\ -1 & 1 & 1 & 0 \\ -4 & 2 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ -2 & 1 & 1 & 0 \end{bmatrix}$$

satisfies $XA = Y$.

Problem 8. Consider the $EA = R$ factorization and the vector $\boldsymbol{b}$ given by

$$\overset{E}{\begin{bmatrix} 1 & -2 & 2 & 1 & 0 \\ 0 & 1 & -2 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 1 & -1 \\ -1 & 1 & -2 & 0 & -1 \end{bmatrix}} \overset{A}{\begin{bmatrix} 0 & 0 & 0 & 1 & -4 & 1 \\ -1 & 6 & 2 & 0 & 5 & 1 \\ -1 & * & 2 & 0 & 5 & 0 \\ 1 & -6 & -2 & -1 & -1 & 1 \\ 1 & -6 & -2 & -1 & -1 & 0 \end{bmatrix}} = \overset{R}{\begin{bmatrix} 1 & -6 & -2 & 0 & -5 & 0 \\ 0 & 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}} \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Note that the $(3,2)$ entry of A is unknown and marked as $*$.

(2 pts) (*a*) Which columns of A are the *pivot columns*? Select all that apply (no partial credit).

√ 1st column ◯ 2nd column ◯ 3rd column √ 4th column ◯ 5th column √ 6th column

(2 pts) (*b*) The unknown $(3,2)$ entry of A is $* =$ __6__.

(4 pts) (*c*) In class we demonstrated that representing the steps of the Gauß-Jordan algorithm with elementary matrices allows E to be expressed as the product of elementary matrices $E = E_k \cdots E_3 E_2 E_1$. Find E_1.

Solution. The first step of the Gauß-Jordan algorithm in row-reducing A is $\boldsymbol{r}_1 \leftrightarrow \boldsymbol{r}_2$. Since A is 5×6, E_1 is the matrix obtained by applying $\boldsymbol{r}_1 \leftrightarrow \boldsymbol{r}_2$ to I_5. This gives

$$\overset{I_5}{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}} \xrightarrow{\boldsymbol{r}_1 \leftrightarrow \boldsymbol{r}_2} \overset{E_1}{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}$$

(5 pts) (*d*) Determine if the system $A\boldsymbol{x} = \boldsymbol{b}$ is consistent. Clearly explain your reasoning to receive credit.

Solution. We discussed in class that the system $[A \mid \boldsymbol{b}]$ reduces to $[R \mid E\boldsymbol{b}]$, which is

$$\left[\begin{array}{cccccc|c} 1 & -6 & -2 & 0 & -5 & 0 & -1 \\ 0 & 0 & 0 & 1 & -4 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

There is a pivot in the augmented column, so the system $A\boldsymbol{x} = \boldsymbol{b}$ is *inconsistent*.

Problem 9. Consider the $PA = LU$ factorization and the vector $\boldsymbol{b}$ given by

$$\overset{P}{\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}} \overset{A}{\begin{bmatrix} -3 & -44 & -16 & 14 & 35 & -5 \\ 0 & 0 & 0 & 0 & -8 & -7 \\ 0 & 0 & 0 & -5 & -73 & 3 \\ 0 & -13 & -3 & -14 & 39 & -1 \\ 3 & 57 & 19 & 5 & 7 & 10 \end{bmatrix}} = \overset{L}{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}} \overset{U}{\begin{bmatrix} 3 & 57 & 19 & 5 & 7 & 10 \\ 0 & 13 & 3 & 19 & 42 & 5 \\ 0 & 0 & 0 & 5 & 81 & 4 \\ 0 & 0 & 0 & 0 & 8 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}} \qquad \boldsymbol{b} = \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

(3 pts) (a) In the system $A\boldsymbol{x} = \boldsymbol{O}$, which of the following variables are *free*? Select all that apply (no partial credit).

○ x_1 ○ x_2 √ x_3 ○ x_4 ○ x_5 √ x_6

(6 pts) (b) Determine if the system $A\boldsymbol{x} = \boldsymbol{b}$ is consistent. Clearly explain your reasoning to receive credit.

Solution. We demonstrated in class that the system $A\boldsymbol{x} = \boldsymbol{b}$ is solved by first solving $L\boldsymbol{y} = P\boldsymbol{b}$ for $\boldsymbol{y}$, which is

$$\begin{aligned} y_1 &= 0 \\ -y_1 + y_2 &= -3 \leftarrow y_2 = -3 \\ -y_2 + y_3 &= 0 \leftarrow y_3 = -3 \\ -y_3 + y_4 &= 1 \leftarrow y_4 = -2 \\ -y_4 + y_5 &= -1 \leftarrow y_5 = -3 \end{aligned}$$

This gives $\boldsymbol{y} = \begin{bmatrix} 0 & -3 & -3 & -2 & -3 \end{bmatrix}^\intercal$. In augmented form, system $U\boldsymbol{x} = \boldsymbol{y}$ is then

$$\left[\begin{array}{cccccc|c} 3 & 57 & 19 & 5 & 7 & 10 & 0 \\ 0 & 13 & 3 & 19 & 42 & 5 & -3 \\ 0 & 0 & 0 & 5 & 81 & 4 & -3 \\ 0 & 0 & 0 & 0 & 8 & 7 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3 \end{array}\right]$$

There is a pivot in the augmented column, so our system is *inconsistent*.

Problem 10. The data below depicts the inverse of a matrix A and a vector $\boldsymbol{b}$.

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & -3 & -2 \\ 0 & -1 & 0 & 3 & -1 \\ -1 & 0 & 0 & 0 & -3 \\ 1 & 0 & 0 & 1 & 3 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

(3 pts) (a) The matrix A has $\operatorname{rank}(A) = \underline{\quad 5 \quad}$, $\operatorname{nullity}(A) = \underline{\quad 0 \quad}$, and $\operatorname{nullity}(A^\intercal) = \underline{\quad 0 \quad}$.

(4 pts) (b) Find all solutions to the system $A\boldsymbol{x} = \boldsymbol{b}$. Clearly explain your work to receive credit.

Solution. The matrix A is invertible so $A\boldsymbol{x} = \boldsymbol{b}$ has exactly one solution given by

$$\boldsymbol{x} = \overset{A^{-1}}{\begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & -3 & -2 \\ 0 & -1 & 0 & 3 & -1 \\ -1 & 0 & 0 & 0 & -3 \\ 1 & 0 & 0 & 1 & 3 \end{bmatrix}} \overset{\boldsymbol{b}}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

(4 pts) (c) Find the second column of A. Clearly explain your work to receive credit.

Solution. This is the same as asking for the value of

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We are given A^{-1}, which is defined by the property

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \overset{A^{-1}}{\begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & -3 & -2 \\ 0 & -1 & 0 & 3 & -1 \\ -1 & 0 & 0 & 0 & -3 \\ 1 & 0 & 0 & 1 & 3 \end{bmatrix}} = \overset{I_5}{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}$$

The third column of A^{-1} is $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix}^\intercal$, which tells us that

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$