Duke University

Math 218D-2

Matrices and Vectors

Exam I

Name: ____________________ *Unique ID:* ____________

I have adhered to the Duke Community Standard in completing this exam.

Signature: ____________________

July 18, 2024

- There are 100 points and 10 problems on this 100-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

Problem 1. Suppose that A is a matrix satisfying the matrix-vector product $\begin{bmatrix} & A & \end{bmatrix}\overset{\boldsymbol{v}}{\begin{bmatrix}-1\\-1\\0\\1\end{bmatrix}} = \begin{bmatrix}1\\-2\\-1\\0\\-1\end{bmatrix}$.

(2 pts) (*a*) The matrix A has _______ rows and _______ columns.

(3 pts) (*b*) Which of the following vectors is *orthogonal* to the vector $\boldsymbol{v}$?

○ 1st row of A ○ 2nd row of A ○ 3rd row of A ○ 4th row of A ○ 3rd column of A

(3 pts) (*c*) Which of the following vectors forms an *acute* angle with $\boldsymbol{v}$?

○ 1st row of A ○ 2nd row of A ○ 3rd row of A ○ 4th row of A ○ 3rd column of A

(4 pts) (*d*) Fill in the entries of this matrix product: $\begin{bmatrix} & A & \end{bmatrix}\begin{bmatrix}1 & 2 & 0 & 2\\1 & 2 & 0 & 2\\0 & 0 & 0 & 0\\-1 & -2 & 0 & -2\end{bmatrix} = \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}$

(4 pts) (*e*) Calculate $\langle A^\intercal \boldsymbol{w}, \boldsymbol{v}\rangle$ where $\boldsymbol{w} = \begin{bmatrix}1 & 1 & 1 & 1 & 1\end{bmatrix}^\intercal$. Clearly explain your reasoning to receive credit.

(6 pts) **Problem 2.** Let $S = A^\intercal A + BCB$ where A is a 5×2024 matrix and B and C are 2024×2024 symmetric matrices. Show that S is symmetric.

(5 pts) **Problem 3.** Suppose that $\boldsymbol{v}$, $\boldsymbol{w}_1$, and $\boldsymbol{w}_2$ are vectors in $\mathbb{R}^n$ such that $\boldsymbol{v}$ is orthogonal to both $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$. Let $\boldsymbol{w} = c_1 \cdot \boldsymbol{w}_1 + c_2 \cdot \boldsymbol{w}_2$. Show that $\boldsymbol{v}$ is orthogonal to $\boldsymbol{w}$.

You must avoid circular reasoning to receive credit.

(5 pts) **Problem 4.** Suppose that A is $n \times n$ and that $\boldsymbol{v} \in \mathcal{E}_A(\lambda)$ and let $X_\lambda = \lambda \cdot I_n - A$. Calculate $X_\lambda \boldsymbol{v}$. Clearly explain your reasoning to receive credit.

(10 pts) **Problem 5.** Consider the matrix A given by

$$A = \begin{bmatrix} 2 & -2 & 2 & 4 & 4 & 6 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 1 & -1 & 1 & 2 & 3 & 4 \\ 0 & -2 & 2 & 2 & 0 & 2 \\ 0 & 1 & -1 & -1 & 0 & -1 \end{bmatrix}$$

Use the Gauß-Jordan algorithm to calculate $\text{rref}(A)$.

You must label each row reduction properly and adhere to the steps of the algorithm to receive credit.

(10 pts) **Problem 6.** Calculate $PA = LU$ for $A = \begin{bmatrix} 2 & 5 & 3 & 1 & 3 \\ 0 & 0 & 0 & 3 & 5 \\ -2 & -2 & 2 & -4 & 2 \\ 4 & 7 & 1 & 2 & 7 \end{bmatrix}$.

Problem 7. Consider the matrices A, B, and Y given by

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 2 & 1 & 3 \\ -2 & -2 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & -1 & 0 & 1 \\ -2 & 1 & 0 & -1 \\ -2 & 1 & 1 & 0 \end{bmatrix} \qquad Y = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(5 pts) (a) Calculate AB. The matrix AB is ________ $\times$ ________ and trace(AB) = ________.

(5 pts) (b) Calculate BA. The matrix BA is ________ $\times$ ________ and trace(BA) = ________.

(5 pts) (c) If possible, find a matrix X satisfying $XA = Y$. If this is not possible, then explain why.

Problem 8. Consider the $EA = R$ factorization and the vector $\boldsymbol{b}$ given by

$$\overset{E}{\begin{bmatrix} 1 & -2 & 2 & 1 & 0 \\ 0 & 1 & -2 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 1 & -1 \\ -1 & 1 & -2 & 0 & -1 \end{bmatrix}} \overset{A}{\begin{bmatrix} 0 & 0 & 0 & 1 & -4 & 1 \\ -1 & 6 & 2 & 0 & 5 & 1 \\ -1 & * & 2 & 0 & 5 & 0 \\ 1 & -6 & -2 & -1 & -1 & 1 \\ 1 & -6 & -2 & -1 & -1 & 0 \end{bmatrix}} = \overset{R}{\begin{bmatrix} 1 & -6 & -2 & 0 & -5 & 0 \\ 0 & 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}} \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Note that the $(3, 2)$ entry of A is unknown and marked as $*$.

(2 pts) (a) Which columns of A are the *pivot columns*? Select all that apply (no partial credit).

○ 1st column ○ 2nd column ○ 3rd column ○ 4th column ○ 5th column ○ 6th column

(2 pts) (b) The unknown $(3, 2)$ entry of A is $* =$ ______.

(4 pts) (c) In class we demonstrated that representing the steps of the Gauß-Jordan algorithm with elementary matrices allows E to be expressed as the product of elementary matrices $E = E_k \cdots E_3 E_2 E_1$. Find E_1.

(5 pts) (d) Determine if the system $A\boldsymbol{x} = \boldsymbol{b}$ is consistent. Clearly explain your reasoning to receive credit.

Problem 9. Consider the $PA = LU$ factorization and the vector $\boldsymbol{b}$ given by

$$\overset{P}{\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}} \overset{A}{\begin{bmatrix} -3 & -44 & -16 & 14 & 35 & -5 \\ 0 & 0 & 0 & 0 & -8 & -7 \\ 0 & 0 & 0 & -5 & -73 & 3 \\ 0 & -13 & -3 & -14 & 39 & -1 \\ 3 & 57 & 19 & 5 & 7 & 10 \end{bmatrix}} = \overset{L}{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}} \overset{U}{\begin{bmatrix} 3 & 57 & 19 & 5 & 7 & 10 \\ 0 & 13 & 3 & 19 & 42 & 5 \\ 0 & 0 & 0 & 5 & 81 & 4 \\ 0 & 0 & 0 & 0 & 8 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}} \qquad \boldsymbol{b} = \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

(3 pts) (a) In the system $A\boldsymbol{x} = \boldsymbol{O}$, which of the following variables are *free*? Select all that apply (no partial credit).

○ x_1 ○ x_2 ○ x_3 ○ x_4 ○ x_5 ○ x_6

(6 pts) (b) Determine if the system $A\boldsymbol{x} = \boldsymbol{b}$ is consistent. Clearly explain your reasoning to receive credit.

Problem 10. The data below depicts the inverse of a matrix A and a vector $\boldsymbol{b}$.

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & -3 & -2 \\ 0 & -1 & 0 & 3 & -1 \\ -1 & 0 & 0 & 0 & -3 \\ 1 & 0 & 0 & 1 & 3 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

(3 pts) (*a*) The matrix A has $\operatorname{rank}(A) =$ _______, $\operatorname{nullity}(A) =$ _______, and $\operatorname{nullity}(A^\intercal) =$ _______.

(4 pts) (*b*) Find all solutions to the system $A\boldsymbol{x} = \boldsymbol{b}$. Clearly explain your work to receive credit.

(4 pts) (*c*) Find the second column of A. Clearly explain your work to receive credit.