Duke University

Math 218D-2

Matrices and Vectors

Exam II

Name: Solutions

Unique ID:

I have adhered to the Duke Community Standard in completing this exam.

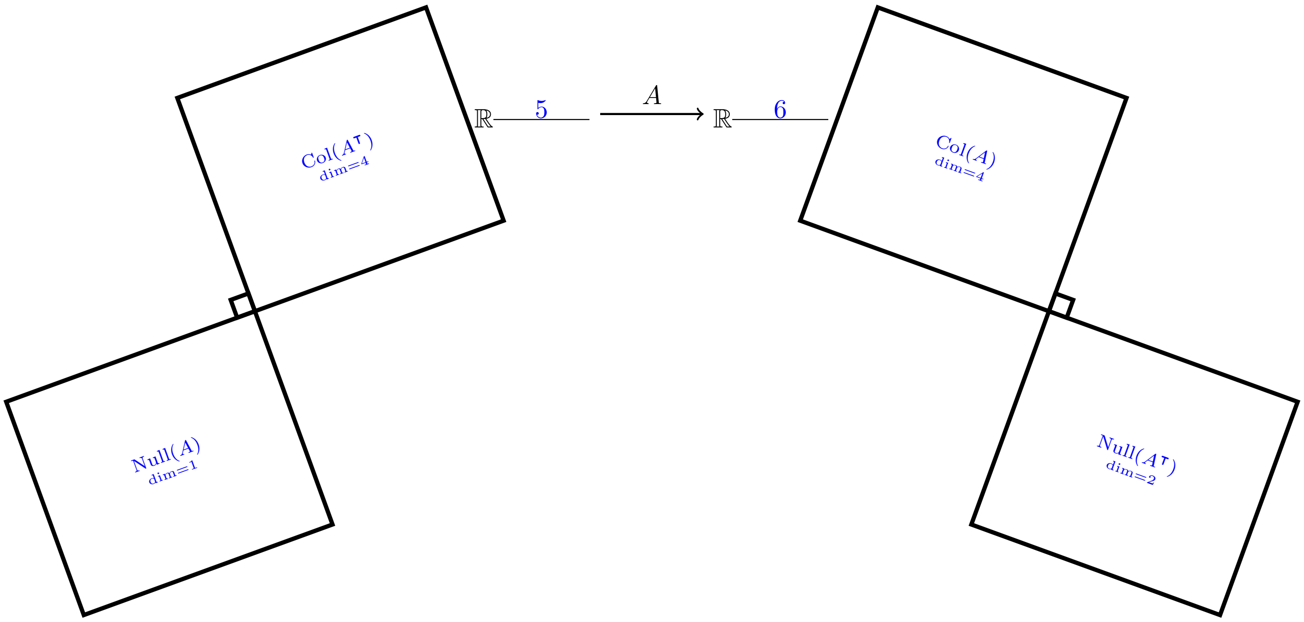
Signature:

August 1, 2024

- There are 100 points and 11 problems on this 100-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

Problem 1. Let A be a matrix satisfying $\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and $\operatorname{rref}(A^\intercal) = \begin{bmatrix} 1 & -3 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(10 pts) (*a*) Fill in every missing label in the picture of the four fundamental subspaces of A below, including the dimension of each fundamental subspace.



(3 pts) (*b*) Which of the following is the most accurate geometric description of the *left null space* of A?

√ a plane in $\mathbb{R}^6$ ○ a plane in $\mathbb{R}^5$ ○ a line in $\mathbb{R}^6$ ○ a line in $\mathbb{R}^5$ ○ a point with six coordinates

(3 pts) (*c*) The projection matrix onto the *row space* of A has trace __4__.

(3 pts) (*d*) Every vector in Null(A) is guaranteed to be orthogonal to only one of the following vectors. Select this vector.

√ $\begin{bmatrix} 1 & 0 & 0 & 4 & 0 \end{bmatrix}^\intercal$ ○ $\begin{bmatrix} 5 & -3 & 3 & 3 & 0 \end{bmatrix}^\intercal$ ○ $\begin{bmatrix} 4 & -2 & 3 & -4 & 0 \end{bmatrix}^\intercal$ ○ $\begin{bmatrix} 4 & -3 & 4 & 2 & 0 \end{bmatrix}^\intercal$

(5 pts) (*e*) Select all of the following vectors belonging to the row space of A (1.25pts each).

√ $\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \end{bmatrix}^\intercal$ ○ $\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \end{bmatrix}^\intercal$ ○ $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}^\intercal$ √ $\begin{bmatrix} 1 & 1 & 1 & 4 & 0 \end{bmatrix}^\intercal$

Solution. This is a question of whether or not each vector is orthogonal to Null(A), which is reasonable to determine by hand because $\dim \operatorname{Null}(A) = 1$. In the system $A\boldsymbol{x} = \boldsymbol{O}$, the free variable is $x_4 = c_1$ and the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -4\,c_1 \\ 3\,c_1 \\ -3\,c_1 \\ c_1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} -4 \\ 3 \\ -3 \\ 1 \\ 0 \end{bmatrix}$$

Our basis vector of Null(A) is $\begin{bmatrix} -4 & 3 & -3 & 1 & 0 \end{bmatrix}^\intercal$. The correct options are the ones orthogonal to this vector.

(5 pts) **Problem 2.** Suppose that $\boldsymbol{v} \in \mathcal{E}_A(5)$ where A is $n \times n$ and let $M = 2\,A^2 - A + I_n$. Show that $\boldsymbol{v} \in \mathcal{E}_M(\lambda)$ and correctly fill in the blank: $\lambda =$ __46__.

Solution. We are told that $\boldsymbol{v} \in \mathcal{E}_A(5)$, which means that $A\boldsymbol{v} = 5 \cdot \boldsymbol{v}$. We wish to find the value of λ that validates the equation $M\boldsymbol{v} = \lambda \cdot \boldsymbol{v}$. To do so, note that

$$\begin{aligned} M\boldsymbol{v} &= (2\,A^2 - A + I_n)\boldsymbol{v} \\ &= 2\,AA\boldsymbol{v} - A\boldsymbol{v} + \boldsymbol{v} \\ &= 2\,A(5 \cdot \boldsymbol{v}) - 5 \cdot \boldsymbol{v} + \boldsymbol{v} \\ &= (2 \cdot 5) \cdot A\boldsymbol{v} - 5 \cdot \boldsymbol{v} + \boldsymbol{v} \\ &= 2 \cdot 5 \cdot 5 \cdot \boldsymbol{v} - 5 \cdot \boldsymbol{v} + \boldsymbol{v} \\ &= 50 \cdot \boldsymbol{v} - 5 \cdot \boldsymbol{v} + \boldsymbol{v} \\ &= 46 \cdot \boldsymbol{v} \end{aligned}$$

This demonstrates that $\boldsymbol{v} \in \mathcal{E}_M(46)$.

(6 pts) **Problem 3.** The only eigenvalue of $A = \begin{bmatrix} 31 & -29 & -19 & -4 \\ 35 & -33 & -23 & -5 \\ -20 & 20 & 15 & 3 \\ 52 & -52 & -33 & -5 \end{bmatrix}$ is $\lambda = 2$ and $\operatorname{gm}_A(2) = 1$. Use this information to find a basis of $\mathcal{E}_A(2)$.

Solution. By definition, $\mathcal{E}_A(2) = \operatorname{Null}(2 \cdot I_4 - A)$. We are told that $\operatorname{gm}_A(2) = 1$, so we expect only one column relation in $2 \cdot I_4 - A$. Indeed, we find

$$\operatorname{rref}\overbrace{\begin{bmatrix} -29 & 29 & 19 & 4 \\ -35 & 35 & 23 & 5 \\ 20 & -20 & -13 & -3 \\ -52 & 52 & 33 & 7 \end{bmatrix}}^{2 \cdot I_4 - A} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We row-reduced $2 \cdot I_4 - A$ by noticing that the second column is the negation of the first. There are no other column relations because $\operatorname{gm}_A(2) = 1$. The space $\mathcal{E}_A(2)$ has a single basis vector found by producing the general solution to $(2 \cdot I_4 - A)\boldsymbol{x} = \boldsymbol{O}$, which gives

$$\mathcal{E}_A(2) = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right\}$$

(6 pts) **Problem 4.** If possible, construct a matrix A with $\begin{bmatrix}1 & -3 & 2\end{bmatrix}^\intercal$ in the row space of A and $\begin{bmatrix}3 & 2 & 2\end{bmatrix}^\intercal$ in the null space of A.

Solution. This is asking for $\begin{bmatrix}1 & -3 & 2\end{bmatrix}^\intercal \in \operatorname{Col}(A^\intercal)$ and $\begin{bmatrix}3 & 2 & 2\end{bmatrix}^\intercal \in \operatorname{Null}(A)$. This is impossible because the inner product of these two vectors is

$$\langle \begin{bmatrix}1 & -3 & 2\end{bmatrix}^\intercal, \begin{bmatrix}3 & 2 & 2\end{bmatrix}^\intercal \rangle = 1 \neq 0$$

The vectors are not orthogonal. The existence of such a matrix A would violate the fact that $\operatorname{Col}(A^\intercal) \perp \operatorname{Null}(A)$.

(6 pts) **Problem 5.** Consider $\overset{E}{\begin{bmatrix}2 & -1 & 2 & -1 & -3\\ 1 & 0 & 2 & -3 & 1\\ 1 & 0 & 3 & -4 & 0\\ 1 & -1 & 0 & 2 & -3\\ -1 & 0 & 1 & -1 & 0\end{bmatrix}} \overset{A}{\begin{bmatrix}-1 & 0 & 2 & 1 & 1 & -5\\ -6 & -3 & 3 & 7 & 5 & -27\\ -5 & 0 & 10 & 4 & 5 & -22\\ -4 & 0 & 8 & 3 & 4 & -17\\ -1 & 1 & 5 & 0 & 1 & -3\end{bmatrix}} = \overset{R}{\begin{bmatrix}1 & 0 & -2 & 0 & 0 & -1\\ 0 & 1 & 3 & 0 & 0 & -1\\ 0 & 0 & 0 & 1 & 0 & -3\\ 0 & 0 & 0 & 0 & 1 & -3\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}}$. Calculate the projection of $\boldsymbol{v} = \begin{bmatrix}-3 & 3 & 3 & -3 & 3\end{bmatrix}^\intercal$ onto the left null space of A.

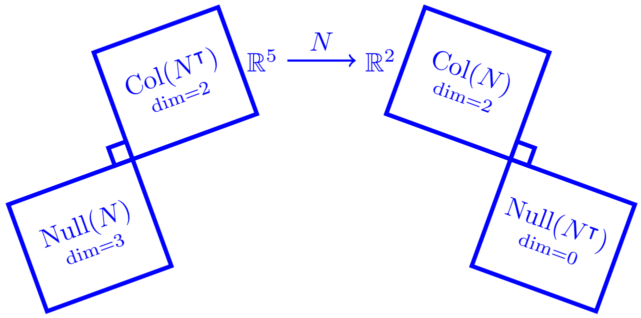
Solution. There is one row of zeros at the bottom of R, so the left null space of A is one-dimensional. We learned in class that the last row of E ($\boldsymbol{v}_1 = \begin{bmatrix}-1 & 0 & 1 & -1 & 0\end{bmatrix}^\intercal$) is a basis vector of $\operatorname{Null}(A)$ in this case. Appealing to the one-dimensional projection formula then gives our desired projection as

$$\frac{1}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1\boldsymbol{v}_1^\intercal\boldsymbol{v} = \frac{1}{3}\begin{bmatrix}-1\\ 0\\ 1\\ -1\\ 0\end{bmatrix}\begin{bmatrix}-1 & 0 & 1 & -1 & 0\end{bmatrix}\begin{bmatrix}-3\\ 3\\ 3\\ -3\\ 3\end{bmatrix} = \begin{bmatrix}-3\\ 0\\ 3\\ -3\\ 0\end{bmatrix}$$

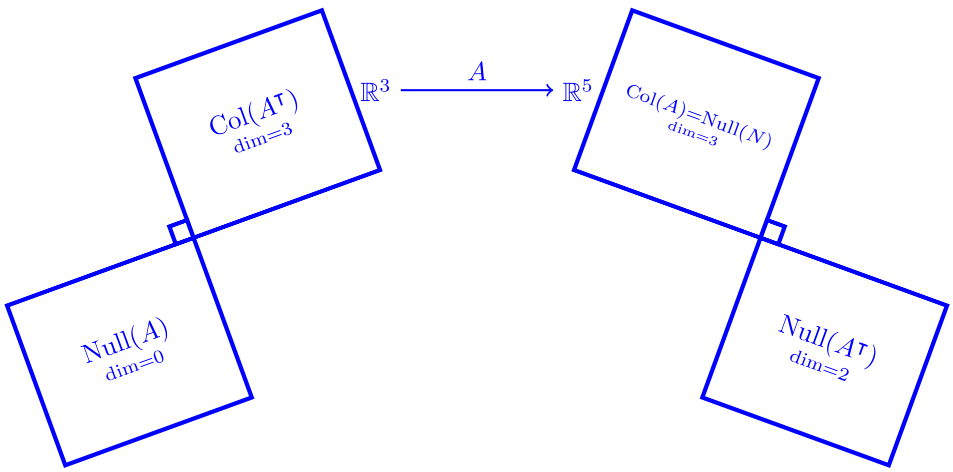
Problem 6. Let A be a matrix with independent columns satisfying $\operatorname{Col}(A) = \operatorname{Null}(N)$ where $N = \begin{bmatrix} 1 & 0 & 1 & 2 & 2 \\ 0 & 1 & 0 & -3 & 4 \end{bmatrix}$.

(6 pts) (*a*) A is $\underline{\quad 5 \quad} \times \underline{\quad 3 \quad}$ with $\operatorname{rank}(A) = \underline{\quad 3 \quad}$. Clearly explain your reasoning below.

Solution. The most straightforward way to do this is to start with the picture of the four fundamental subspaces of N, noting that N is in reduced row echelon form.



The picture for A must then be



The right side of the picture is inherited from the left side of the picture of N. Since A has independent columns, we then fill in the left side of the picture by accounting for the fact that A must be *full column rank.*

(4 pts) (*b*) Let $\boldsymbol{b} = \begin{bmatrix} 0 & 4 & 2 & 0 & -1 \end{bmatrix}^\intercal$. Is the system $A\boldsymbol{x} = \boldsymbol{b}$ consistent? Clearly explain why or why not.

Solution. This is the same as asking if $\boldsymbol{b} \in \operatorname{Col}(A)$. Since $\operatorname{Col}(A) = \operatorname{Null}(N)$, we can quickly resolve the issue by calculating $N\boldsymbol{b}$.

$$\overset{N}{\begin{bmatrix} 1 & 0 & 1 & 2 & 2 \\ 0 & 1 & 0 & -3 & 4 \end{bmatrix}} \overset{\boldsymbol{b}}{\begin{bmatrix} 0 \\ 4 \\ 2 \\ 0 \\ -1 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Here we find that $N\boldsymbol{b} = \boldsymbol{O}$, so $\boldsymbol{b} \in \operatorname{Null}(N) = \operatorname{Col}(A)$. This means that $A\boldsymbol{x} = \boldsymbol{b}$ is, in fact, consistent.

Problem 7. Consider the matrix A and the vector $\boldsymbol{b}$ given by

$$A = \begin{bmatrix} 1 & 4 & -4 \\ 1 & 0 & -2 \\ 2 & -1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ -3 \\ 2 \\ -1 \end{bmatrix}$$

The solution $\widehat{\boldsymbol{x}}$ to the least squares problem associated to $A\boldsymbol{x} = \boldsymbol{b}$ is $\widehat{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^\intercal$.

(4 pts) (a) Calculate the error E in using the least squares technique to approximate a solution to $A\boldsymbol{x} = \boldsymbol{b}$.

Solution. This is

$$E = \left\| \overset{\boldsymbol{b}}{\begin{bmatrix} 1 \\ -3 \\ 2 \\ -1 \end{bmatrix}} - \overset{A}{\begin{bmatrix} 1 & 4 & -4 \\ 1 & 0 & -2 \\ 2 & -1 & 1 \\ 2 & 1 & 0 \end{bmatrix}} \overset{\widehat{\boldsymbol{x}}}{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}} \right\|^2 = \left\| \begin{bmatrix} 1 \\ -3 \\ 2 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix} \right\|^2 = 10$$

(5 pts) (b) Find the projection of $\boldsymbol{b}$ onto $\operatorname{Null}(A^\intercal)$.

Solution. The projection of $\boldsymbol{b}$ onto $\operatorname{Col}(A)$ is

$$\overset{A}{\begin{bmatrix} 1 & 4 & -4 \\ 1 & 0 & -2 \\ 2 & -1 & 1 \\ 2 & 1 & 0 \end{bmatrix}} \overset{\widehat{x}}{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}} = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

The orthogonality of $\operatorname{Col}(A)$ and $\operatorname{Null}(A^\intercal)$ then tells us that the projection of $\boldsymbol{b}$ onto $\operatorname{Null}(A^\intercal)$ is

$$\overset{\boldsymbol{b}}{\begin{bmatrix} 1 \\ -3 \\ 2 \\ -1 \end{bmatrix}} - \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix}$$

(6 pts) **Problem 8.** Suppose that Q_1 and Q_2 have orthonormal columns. Show that $Q = Q_1 Q_2$ also has orthonormal columns.

Solution. We need only check $Q^\intercal Q = I$, which follows from

$$Q^\intercal Q = (Q_1 Q_2)^\intercal (Q_1 Q_2) = Q_2^\intercal Q_1^\intercal Q_1 Q_2 = Q_2^\intercal I Q_2 = Q_2^\intercal Q_2 = I$$

(6 pts) **Problem 9.** Suppose that an $n \times n$ matrix P is *symmetric* and *idempotent* and let $\boldsymbol{v} \in \mathbb{R}^n$. Show that the vectors $P\boldsymbol{v}$ and $(I - P)\boldsymbol{v}$ are orthogonal.

Solution. We are told that P is *symmetric* and *idempotent*, which means that $P^\intercal = P$ and $P^2 = P$. We wish to demonstrate that $P\boldsymbol{v}$ and $(I - P)\boldsymbol{v}$ are orthogonal. To do so, note that

$$\begin{aligned}
\langle P\boldsymbol{v}, (I_n - P)\boldsymbol{v}\rangle &= \langle \boldsymbol{v}, P^\intercal (I_n - P)\boldsymbol{v}\rangle \\
&= \langle \boldsymbol{v}, P(I_n - P)\boldsymbol{v}\rangle \\
&= \langle \boldsymbol{v}, (P - P^2)\boldsymbol{v}\rangle \\
&= \langle \boldsymbol{v}, (P - P)\boldsymbol{v}\rangle \\
&= \langle \boldsymbol{v}, \boldsymbol{O}_n\boldsymbol{v}\rangle \\
&= 0
\end{aligned}$$

Problem 10. Let A be the incidence matrix of a directed graph G such that $A = QR$ where

$$Q = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R = \sqrt{2}\begin{bmatrix} 1 & 0 & 1 & 0 & -1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Do not ignore the factor $1/\sqrt{2}$ used to define Q and the factor $\sqrt{2}$ used to define R!

(6 pts) (*a*) $\chi(G) =$ __-2__, $h_0(G) =$ __3__, and $h_1(G) =$ __5__

Solution. The $A = QR$ factorization tells us that A is 6×8, which is why $\chi(G) = 6 - 8 = -2$. The number of columns of Q is the rank of A, which means that $\operatorname{rank}(A) = 3$. It follows that $h_0(G) = \operatorname{nullity}(A^\intercal) = 6 - 3 = 3$ and $h_1(G) = \operatorname{nullity}(A) = 8 - 3 = 5$.

(6 pts) (*b*) Is it possible to set weights on the arrows of G so that the net flow through the nodes of G is given by the vector $\boldsymbol{b} = \begin{bmatrix} 2 & -2 & 0 & 0 & 0 & 0 \end{bmatrix}^\intercal$? Clearly explain why or why not.

Solution. This is the same thing as asking whether or not there is a weight vector $\boldsymbol{w}$ satisfying $A\boldsymbol{w} = \boldsymbol{b}$, which is the same question of whether or not $\boldsymbol{b} \in \operatorname{Col}(A)$.

Given our $A = QR$ factorization, the quickest way to sort out this issue is to see if $\boldsymbol{b}$ is stable upon projection to $\operatorname{Col}(A)$. Recall that $P_{\operatorname{Col}(A)} = QQ^\intercal$, so or question is resolved with the calculation

$$\overbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{Q} \overbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}}^{Q^\intercal} \overbrace{\begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}^{\boldsymbol{b}} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

Here, we find $QQ^\intercal\boldsymbol{b} \neq \boldsymbol{b}$, so $\boldsymbol{b} \notin \operatorname{Col}(A)$. This means that is is *not possible* to set weights on the arrows of G so that the net flow is given by this vector $\boldsymbol{b}$.

(10 pts)**Problem 11.** Calculate $A = QR$ for $A = \begin{bmatrix} 0 & -1 & 2 \\ 2 & 4 & -8 \\ -1 & -1 & 2 \\ -2 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

Solution. We begin by applying the Gram-Schmidt procedure to the columns of A.

$$\boldsymbol{v}_1 = \begin{bmatrix} 0 & 2 & -1 & -2 & 0 \end{bmatrix}^\intercal \qquad \boldsymbol{v}_2 = \begin{bmatrix} -1 & 4 & -1 & 0 & 0 \end{bmatrix}^\intercal \qquad \boldsymbol{v}_3 = \begin{bmatrix} 2 & -8 & 2 & 0 & 3 \end{bmatrix}^\intercal$$

Of course $\boldsymbol{w}_1 = \boldsymbol{v}_1 = \begin{bmatrix} 0 & 2 & -1 & -2 & 0 \end{bmatrix}^\intercal$ is the simplest. Next, $\boldsymbol{w}_2$ is

$$\boldsymbol{w}_1 = \boldsymbol{v}_2 - \operatorname{proj}_{\boldsymbol{w}_1}(\boldsymbol{v}_2) = \boldsymbol{v}_2 - \frac{\langle \boldsymbol{w}_1, \boldsymbol{v}_2 \rangle}{\langle \boldsymbol{w}_1, \boldsymbol{w}_1 \rangle}\boldsymbol{w}_1 = \begin{bmatrix} -1 \\ 4 \\ -1 \\ 0 \\ 0 \end{bmatrix} - \frac{9}{9}\begin{bmatrix} 0 \\ 2 \\ -1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

Then $\boldsymbol{w}_3$ is

$$\boldsymbol{w}_3 = \boldsymbol{v}_3 - \operatorname{proj}_{\boldsymbol{w}_1}(\boldsymbol{v}_3) - \operatorname{proj}_{\boldsymbol{w}_2}(\boldsymbol{v}_3) = \boldsymbol{v}_3 - \frac{\langle \boldsymbol{w}_1, \boldsymbol{v}_3 \rangle}{\langle \boldsymbol{w}_1, \boldsymbol{w}_1 \rangle}\boldsymbol{w}_1 - \frac{\langle \boldsymbol{w}_2, \boldsymbol{v}_3 \rangle}{\langle \boldsymbol{w}_2, \boldsymbol{w}_2 \rangle}\boldsymbol{w}_2 = \begin{bmatrix} 2 \\ -8 \\ 2 \\ 0 \\ 3 \end{bmatrix} - \frac{-18}{9}\begin{bmatrix} 0 \\ 2 \\ -1 \\ -2 \\ 0 \end{bmatrix} - \frac{-18}{9}\begin{bmatrix} -1 \\ 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 3 \end{bmatrix}$$

Normalizing $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$ gives $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$ as

$$\boldsymbol{q}_1 = \frac{1}{\sqrt{9}}\begin{bmatrix} 0 \\ 2 \\ -1 \\ -2 \\ 0 \end{bmatrix} \qquad \boldsymbol{q}_2 = \frac{1}{\sqrt{9}}\begin{bmatrix} -1 \\ 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} \qquad \boldsymbol{q}_3 = \frac{1}{\sqrt{9}}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 3 \end{bmatrix}$$

Our desired Q is then

$$Q = \frac{1}{3}\begin{bmatrix} 0 & -1 & 0 \\ 2 & 2 & 0 \\ -1 & 0 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Our desired R is

$$R = \frac{1}{3}\overbrace{\begin{bmatrix} 0 & 2 & -1 & -2 & 0 \\ -1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}}^{Q^\intercal}\overbrace{\begin{bmatrix} 0 & -1 & 2 \\ 2 & 4 & -8 \\ -1 & -1 & 2 \\ -2 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}}^{A} = \frac{1}{3}\begin{bmatrix} 9 & 9 & -18 \\ 0 & 9 & -18 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 3 & 3 & -6 \\ 0 & 3 & -6 \\ 0 & 0 & 3 \end{bmatrix}$$