Duke University

Math 218D-2

Matrices and Vectors

Exam II

Name: ______________________ *Unique ID:* ______________________

I have adhered to the Duke Community Standard in completing this exam.

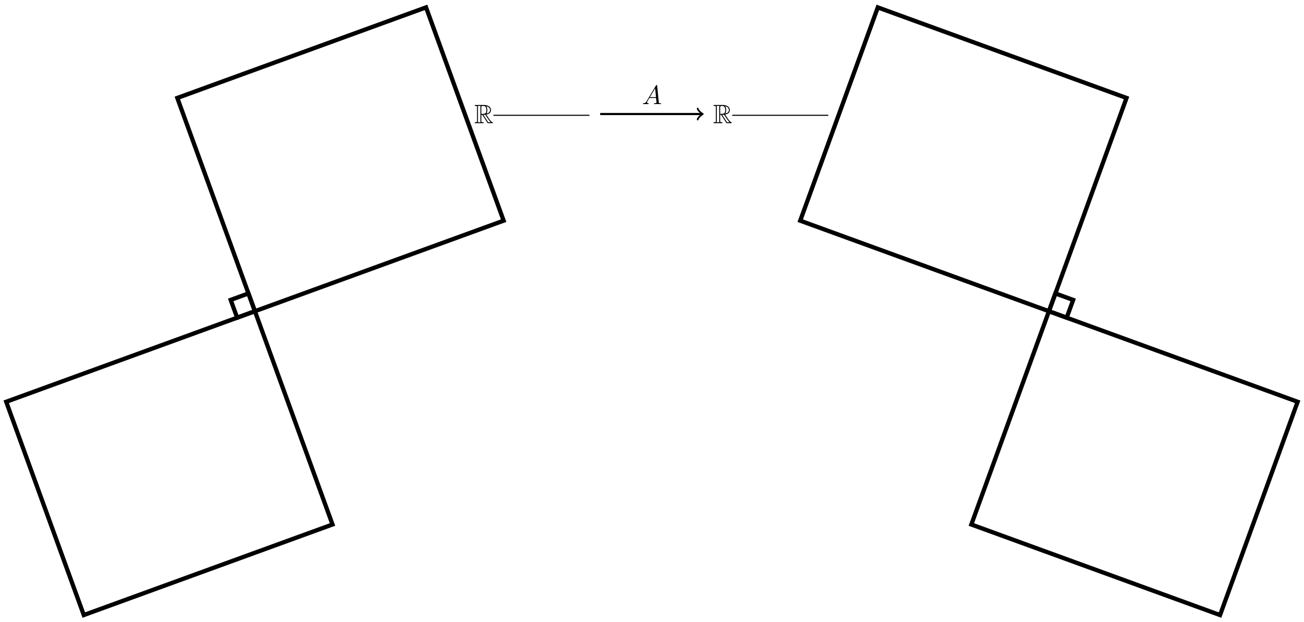
Signature: ______________________

August 1, 2024

- There are 100 points and 11 problems on this 100-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

Problem 1. Let A be a matrix satisfying $\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and $\operatorname{rref}(A^\intercal) = \begin{bmatrix} 1 & -3 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(10 pts) (a) Fill in every missing label in the picture of the four fundamental subspaces of A below, including the dimension of each fundamental subspace.



(3 pts) (b) Which of the following is the most accurate geometric description of the *left null space* of A?

○ a plane in $\mathbb{R}^6$ ○ a plane in $\mathbb{R}^5$ ○ a line in $\mathbb{R}^6$ ○ a line in $\mathbb{R}^5$ ○ a point with six coordinates

(3 pts) (c) The projection matrix onto the *row space* of A has trace ______.

(3 pts) (d) Every vector in Null(A) is guaranteed to be orthogonal to only one of the following vectors. Select this vector.

○ $\begin{bmatrix}1 & 0 & 0 & 4 & 0\end{bmatrix}^\intercal$ ○ $\begin{bmatrix}5 & -3 & 3 & 3 & 0\end{bmatrix}^\intercal$ ○ $\begin{bmatrix}4 & -2 & 3 & -4 & 0\end{bmatrix}^\intercal$ ○ $\begin{bmatrix}4 & -3 & 4 & 2 & 0\end{bmatrix}^\intercal$

(5 pts) (e) Select all of the following vectors belonging to the row space of A (1.25pts each).

○ $\begin{bmatrix}0 & 1 & 1 & 0 & 0\end{bmatrix}^\intercal$ ○ $\begin{bmatrix}1 & 0 & 0 & 0 & 1\end{bmatrix}^\intercal$ ○ $\begin{bmatrix}1 & 0 & 0 & 0 & 0\end{bmatrix}^\intercal$ ○ $\begin{bmatrix}1 & 1 & 1 & 4 & 0\end{bmatrix}^\intercal$

(5 pts) **Problem 2.** Suppose that $\boldsymbol{v} \in \mathcal{E}_A(5)$ where A is $n \times n$ and let $M = 2A^2 - A + I_n$. Show that $\boldsymbol{v} \in \mathcal{E}_M(\lambda)$ and correctly fill in the blank: $\lambda =$ ________.

(6 pts) **Problem 3.** The only eigenvalue of $A = \begin{bmatrix} 31 & -29 & -19 & -4 \\ 35 & -33 & -23 & -5 \\ -20 & 20 & 15 & 3 \\ 52 & -52 & -33 & -5 \end{bmatrix}$ is $\lambda = 2$ and $\operatorname{gm}_A(2) = 1$. Use this information to find a basis of $\mathcal{E}_A(2)$.

(6 pts) **Problem 4.** If possible, construct a matrix A with $\begin{bmatrix}1 & -3 & 2\end{bmatrix}^\intercal$ in the row space of A and $\begin{bmatrix}3 & 2 & 2\end{bmatrix}^\intercal$ in the null space of A.

(6 pts) **Problem 5.** Consider
$$\overset{E}{\begin{bmatrix}2 & -1 & 2 & -1 & -3\\ 1 & 0 & 2 & -3 & 1\\ 1 & 0 & 3 & -4 & 0\\ 1 & -1 & 0 & 2 & -3\\ -1 & 0 & 1 & -1 & 0\end{bmatrix}}\overset{A}{\begin{bmatrix}-1 & 0 & 2 & 1 & 1 & -5\\ -6 & -3 & 3 & 7 & 5 & -27\\ -5 & 0 & 10 & 4 & 5 & -22\\ -4 & 0 & 8 & 3 & 4 & -17\\ -1 & 1 & 5 & 0 & 1 & -3\end{bmatrix}} = \overset{R}{\begin{bmatrix}1 & 0 & -2 & 0 & 0 & -1\\ 0 & 1 & 3 & 0 & 0 & -1\\ 0 & 0 & 0 & 1 & 0 & -3\\ 0 & 0 & 0 & 0 & 1 & -3\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}}.$$
Calculate the projection of $\boldsymbol{v} = \begin{bmatrix}-3 & 3 & 3 & -3 & 3\end{bmatrix}^\intercal$ onto the left null space of A.

Problem 6. Let A be a matrix with independent columns satisfying $\operatorname{Col}(A) = \operatorname{Null}(N)$ where $N = \begin{bmatrix} 1 & 0 & 1 & 2 & 2 \\ 0 & 1 & 0 & -3 & 4 \end{bmatrix}$.

(6 pts) (a) A is ________ $\times$ ________ with $\operatorname{rank}(A) =$ ________. Clearly explain your reasoning below.

(4 pts) (b) Let $\boldsymbol{b} = \begin{bmatrix} 0 & 4 & 2 & 0 & -1 \end{bmatrix}^\intercal$. Is the system $A\boldsymbol{x} = \boldsymbol{b}$ consistent? Clearly explain why or why not.

Problem 7. Consider the matrix A and the vector $\boldsymbol{b}$ given by

$$A = \begin{bmatrix} 1 & 4 & -4 \\ 1 & 0 & -2 \\ 2 & -1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ -3 \\ 2 \\ -1 \end{bmatrix}$$

The solution $\widehat{\boldsymbol{x}}$ to the least squares problem associated to $A\boldsymbol{x} = \boldsymbol{b}$ is $\widehat{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^{\intercal}$.

(4 pts) (*a*) Calculate the error E in using the least squares technique to approximate a solution to $A\boldsymbol{x} = \boldsymbol{b}$.

(5 pts) (*b*) Find the projection of $\boldsymbol{b}$ onto $\operatorname{Null}(A^{\intercal})$.

(6 pts) **Problem 8.** Suppose that Q_1 and Q_2 have orthonormal columns. Show that $Q = Q_1 Q_2$ also has orthonormal columns.

(6 pts) **Problem 9.** Suppose that an $n \times n$ matrix P is *symmetric* and *idempotent* and let $\boldsymbol{v} \in \mathbb{R}^n$. Show that the vectors $P\boldsymbol{v}$ and $(I - P)\boldsymbol{v}$ are orthogonal.

Problem 10. Let A be the incidence matrix of a directed graph G such that $A = QR$ where

$$Q = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R = \sqrt{2}\begin{bmatrix} 1 & 0 & 1 & 0 & -1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Do not ignore the factor $1/\sqrt{2}$ used to define Q and the factor $\sqrt{2}$ used to define R!

(6 pts) (a) $\chi(G) =$ ________, $h_0(G) =$ ________, and $h_1(G) =$ ________

(6 pts) (b) Is it possible to set weights on the arrows of G so that the net flow through the nodes of G is given by the vector $\boldsymbol{b} = \begin{bmatrix} 2 & -2 & 0 & 0 & 0 & 0 \end{bmatrix}^{\intercal}$? Clearly explain why or why not.

(10 pts)**Problem 11.** Calculate $A = QR$ for $A = \begin{bmatrix} 0 & -1 & 2 \\ 2 & 4 & -8 \\ -1 & -1 & 2 \\ -2 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.