Duke University

Math 218D-2

Matrices and Vectors

Exam I

Name: ______________________ *Unique ID:* ______________________

I have adhered to the Duke Community Standard in completing this exam.

Signature: ______________________

May 29, 2025

- There are 100 points and 12 problems on this 100-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

(5 pts) **Problem 1.** Suppose that $S = A + A^\intercal$ where A is 2025×2025. Show that S is *symmetric.* This can be done very quickly by filling in the blanks below.

$$S^\intercal = (A + A^\intercal)^\intercal = \underline{\qquad\qquad\qquad\qquad} = \underline{\qquad\qquad\qquad\qquad} = \underline{\qquad\qquad\qquad\qquad}$$

Problem 2. Calculate each of the following matrix-vector products. If the vector in the product is an eigenvector of the matrix, then fill in the blank with the corresponding eigenvalue. If the vector is not an eigenvector, then select "not an eigenvector".

(2 pts) *(a)* $\begin{bmatrix} 9 & 12 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$ $\lambda =$ ______ ○ not an eigenvector

(2 pts) *(b)* $\begin{bmatrix} 9 & 12 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$ $\lambda =$ ______ ○ not an eigenvector

(2 pts) *(c)* $\begin{bmatrix} 9 & 12 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$ $\lambda =$ ______ ○ not an eigenvector

(2 pts) *(d)* $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$ $\lambda =$ ______ ○ not an eigenvector

(2 pts) *(e)* $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$ $\lambda =$ ______ ○ not an eigenvector

(2 pts) *(f)* $\begin{bmatrix} -11 & -36 & 18 & -18 \\ 3 & 10 & -6 & 6 \\ 3 & 12 & -8 & 6 \\ 3 & 12 & -6 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \\ \quad \end{bmatrix}$ $\lambda =$ ______ ○ not an eigenvector

(2 pts) *(g)* $\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$ $\lambda =$ ______ ○ not an eigenvector

Problem 3. The data below depicts the incidence matrix A of a weighted directed graph G along with the weight vector $\boldsymbol{w}$.

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad \boldsymbol{w} = \begin{bmatrix} 4 \\ 5 \\ -2 \\ 3 \end{bmatrix}$$

Fill in the blanks at the bottom of this page. Use the blank space for any necessary scratch work.

(1 pt) (a) $\chi(G) =$ _______

(4 pts) (b) The number of connected components in G is _______ and the circuit rank of G is _______.

(2 pts) (c) The net flow through the fourth node in G is _______

(5 pts) **Problem 4.** Suppose that $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are vectors in $\mathbb{R}^n$ whose lengths are equal. Let $\boldsymbol{w}_1 = \boldsymbol{v}_1 - \boldsymbol{v}_2$ and let $\boldsymbol{w}_2 = \boldsymbol{v}_1 + \boldsymbol{v}_2$. Show that $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ are orthogonal. Clearly demonstrate your reasoning to receive credit.

(5 pts) **Problem 5.** Consider the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ given by

$$\boldsymbol{v} = \begin{bmatrix}-1 & 2 & -4\end{bmatrix}^\intercal \qquad \boldsymbol{w} = \begin{bmatrix}101 & 1 & -1\end{bmatrix}^\intercal$$

Suppose that A is a 3×3 symmetric matrix satisfying $A\boldsymbol{v} = \begin{bmatrix}0 & 18 & 14\end{bmatrix}^\intercal$. Calculate $\langle A\boldsymbol{w}, \boldsymbol{v}\rangle$. Clearly demonstrate your reasoning to receive credit.

Problem 6. The equation below depicts the result of multiplying two matrices A and B.

$$\overset{A}{\begin{bmatrix} | & | & | & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 \\ | & | & | & | \end{bmatrix}} \overset{B}{\begin{bmatrix} -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 1 & -1 \\ 1 & 1 & 0 & -1 & 1 \end{bmatrix}} = \begin{bmatrix} 0 & 0 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & -1 \end{bmatrix}$$

Note that the columns of A are labeled as $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4$.

(2 pts) (a) A has _______ rows.

(2 pts) (b) $\|\boldsymbol{a}_1 - \boldsymbol{a}_2 + \boldsymbol{a}_3\|^2 =$ _______

(3 pts) (c) Calculate this matrix product: $\begin{bmatrix} | & | & | \\ \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 \\ | & | & | \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ & \\ & \end{bmatrix}$

(4 pts) (d) Find the $(4,4)$ entry of $A^\intercal A$. Clearly demonstrate your reasoning to receive credit. Fill in the blank at the bottom of this page to make your answer clear.

$(4,4)$ entry of $A^\intercal A$ is _______

Problem 7. The following equation shows the result of reducing a system $[A \mid \boldsymbol{b}]$ to reduced row echelon form.

$$\operatorname{rref}\left[\begin{array}{cccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \\ -7 & -3 & 15 & 5 & -28 & 13 & -5 \\ 4 & 5 & -2 & -25 & 7 & -117 & -128 \\ -2 & -7 & -8 & 36 & 2 & 176 & 197 \\ -4 & -2 & 8 & 5 & -15 & 18 & 10 \end{array}\right] = \left[\begin{array}{cccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \\ 1 & 0 & -3 & 0 & 3 & 0 & 1 \\ 0 & 1 & 2 & 0 & 4 & 0 & 7 \\ 0 & 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

(2 pts) (*a*) Columns one, two, four, and six of A are called the ____________________ columns of A.

(2 pts) (*b*) The variables x_1, x_2, x_4, and x_6 are called the ____________________ variables.

(2 pts) (*c*) Only one of the following statements explains why this system is consistent. Select this statement.

○ The system has free variables. ○ $\operatorname{rank}(A) = \operatorname{rank}[A \mid \boldsymbol{b}]$ ○ A is full rank.

○ There are fewer equations than variables. ○ There are no rows of zeros at the bottom of $\operatorname{rref}[A \mid \boldsymbol{b}]$.

(5 pts) (*d*) Some, but not necessarily all, of the following matrices are *full row rank*. Select these matrices (1pt each).

○ A ○ $[A \mid \boldsymbol{b}]$ ○ $A^\intercal A$ ○ A with its third column deleted ○ A with its sixth column deleted

(6 pts) (*e*) Use the technique discussed in class to write the general solution to this system as $\boldsymbol{x} = \boldsymbol{x}_p + c_1 \cdot \boldsymbol{x}_1 + c_2 \cdot \boldsymbol{x}_2$. Clearly explain your reasoning to receive credit. Fill in the blank vectors at the bottom of this page to make your answer clear.

$$\boldsymbol{x} = \overset{\boldsymbol{x}_p}{\begin{bmatrix} \\ \\ \\ \\ \\ \\ \end{bmatrix}} + c_1 \cdot \overset{\boldsymbol{x}_1}{\begin{bmatrix} \\ \\ \\ \\ \\ \\ \end{bmatrix}} + c_2 \cdot \overset{\boldsymbol{x}_2}{\begin{bmatrix} \\ \\ \\ \\ \\ \\ \end{bmatrix}}$$

Problem 8. The following equation shows the result of reducing a system $[A \mid \boldsymbol{b}]$ to *nonreduced* row echelon form.

$$\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \\ 3 & 13 & 29 & -47 & 123 & 150 \\ -15 & -63 & -141 & 228 & -596 & -727 \\ 15 & 63 & 141 & -225 & 593 & 730 \\ 9 & 39 & 87 & -135 & 368 & 466 \\ -9 & -45 & -99 & 165 & -419 & -496 \end{array} \xrightarrow{\text{lots of row operations}} \begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \\ 3 & 3 & 9 & 0 & 1 & 17 \\ 0 & 2 & 4 & 2 & 0 & 12 \\ 0 & 0 & 0 & 3 & 2 & 13 \\ 0 & 0 & 0 & 0 & 5 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

(2 pts) (*a*) $\operatorname{rank}(A) =$ _______ and $\operatorname{nullity}(A) =$ _______

(2 pts) (*b*) Which of the variables in the system is *free*? ○ x_1 ○ x_2 ○ x_3 ○ x_4 ○ x_5

(2 pts) (*c*) Only one of the following statements accurately explains how the concept of "column relations" applies to the matrix A. Select this statement.

○ The third column of A is a multiple of the first column of A. ○ A has no column relations.

○ The third column of A is a linear combination of the first two columns of A.

○ The last column of A is a linear combination of the other four columns of A.

(6 pts) (*d*) Use back-substitution to express the general solution to this system as $\boldsymbol{x} = \boldsymbol{x}_p + c_1 \cdot \boldsymbol{x}_1$. Clearly explain your reasoning to receive credit. Fill in the blank vectors at the bottom of this page to make your answer clear.

$$\boldsymbol{x} = \overset{\boldsymbol{x}_p}{\begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix}} + c_1 \cdot \overset{\boldsymbol{x}_1}{\begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix}}$$

(9 pts) **Problem 9.** Fill-in the blank next to each of the following matrices with the appropriate notation to indicate the first step called for by the Gauß-Jordan algorithm as articulated in class. You do not need to perform the calculation but you must use correct notation to receive credit. (No partial credit. 1.5pts each)

$$\begin{bmatrix} 7 & 5 & 5 & 19 & 3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & -991 & 2 & 9 \\ 0 & 0 & 5 & 8 & 2 \\ 0 & 0 & 1 & 0 & 10 \end{bmatrix} \underline{\hspace{4cm}}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 99999 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \underline{\hspace{4cm}}$$

$$\begin{bmatrix} 1 & 15 & -9 & 8 & 4 \\ 0 & 1 & 3 & 2 & 1 \\ 0 & 0 & 0 & 99 & 8 \\ 0 & 0 & 0 & 1 & 42 \end{bmatrix} \underline{\hspace{4cm}}$$

$$\begin{bmatrix} 11 & 12 & 13 \\ 14 & 15 & 16 \\ 17 & 18 & 19 \\ 20 & 21 & 22 \\ 1 & 23 & 24 \end{bmatrix} \underline{\hspace{4cm}}$$

$$\begin{bmatrix} 1 & 0 & 5 & 15 \\ 0 & 0 & 0 & 1 \\ -7 & 15 & 99 & 87 \end{bmatrix} \underline{\hspace{4cm}}$$

$$\begin{bmatrix} 0 & 0 & 13 & 5 & 1 \\ 0 & 0 & 8 & 3 & 3 \\ 0 & 0 & 1 & 14 & 99 \\ 0 & 0 & 15 & 48 & 6 \end{bmatrix} \underline{\hspace{4cm}}$$

(5 pts) **Problem 10.** Consider the calculation $\operatorname{rref} \overbrace{\begin{bmatrix} -19 & -1 & -1 \\ -5 & 2 & 1 \\ 1 & 5 & 4 \\ 1 & 4 & 1 \\ 0 & 12 & -1 \end{bmatrix}}^{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Does $(A^\intercal A)^{-1}$ exist? *Clearly explain why or why not.*

(5 pts)**Problem 11.** Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & -2 \\ 2 & 2 & 1 \end{bmatrix}$. Calculate A^{-1}. Clearly demonstrate your reasoning to receive credit. Fill in the blank matrix below to make your answer clear.

$$A^{-1} = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}$$

(5 pts)**Problem 12.** Consider the matrix $A = \begin{bmatrix} -278 & 396 & 728 \\ -67 & 95 & 176 \\ -70 & 100 & 183 \end{bmatrix}$ and the vector $\boldsymbol{b} = \begin{bmatrix} 24 \\ 6 \\ 6 \end{bmatrix}$. It is known that

$$A^3 - 7\,A - 6 \cdot I_3 = \boldsymbol{O}_3$$

and that $\boldsymbol{b} \in \mathcal{E}_A(3)$. Use this information to find *all solutions* $\boldsymbol{x}$ to the system $A\boldsymbol{x} = \boldsymbol{b}$. Clearly demonstrate your reasoning to receive credit.