Duke University

Math 218D-2

Matrices and Vectors

Exam II

Name: Solutions

Unique ID:

I have adhered to the Duke Community Standard in completing this exam.

Signature:

June 17, 2025

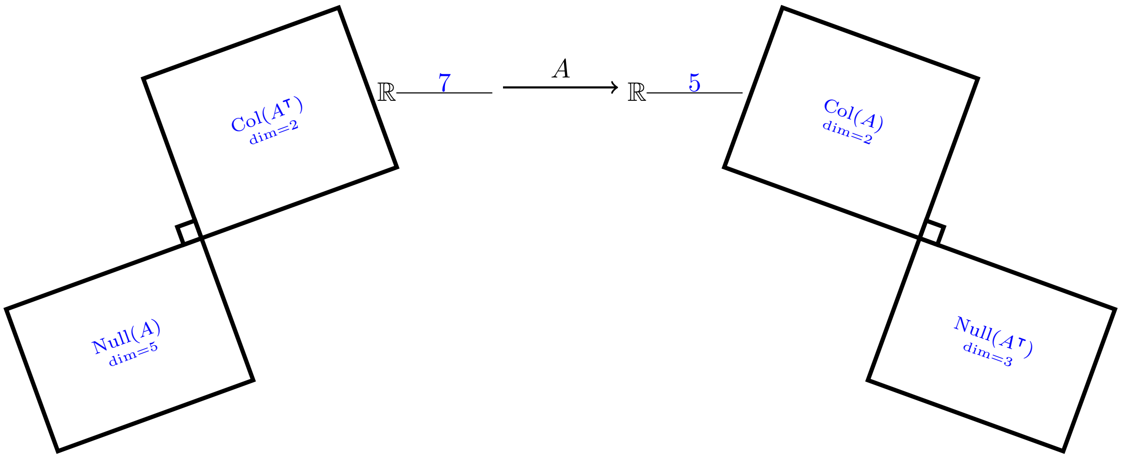
- There are 100 points and 11 problems on this 100-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

Problem 1. Consider the following $EA = R$ factorization.

$$\overset{E}{\begin{bmatrix} 0 & 1 & 2 & 0 & 0 \\ -3 & 1 & -2 & 0 & 1 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & 1 & 2 \\ 2 & 2 & 2 & -2 & 1 \end{bmatrix}} \overset{A}{\begin{bmatrix} 13 & 65 & 5 & 101 & 67 & * & 64 \\ 25 & 125 & 10 & 195 & 130 & * & 125 \\ -12 & -60 & -5 & -94 & -63 & * & -61 \\ 21 & 105 & 8 & 163 & 108 & * & 103 \\ -10 & -50 & -4 & -78 & -52 & * & -50 \end{bmatrix}} = \overset{R}{\begin{bmatrix} 1 & 5 & 0 & 7 & 4 & 1 & 3 \\ 0 & 0 & 1 & 2 & 3 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}$$

Note that the data in the sixth column of A is missing and marked $*$.

(6 pts) (a) Fill in every missing label in the picture of the four fundamental subspaces of A below, including the dimension of each fundamental subspace.



(2 pts) (b) The missing column of A is $\begin{bmatrix} 18 \\ 35 \\ -17 \\ 29 \\ -14 \end{bmatrix}$

(2 pts) (c) Only one of the following vectors belongs to the column space of A. Select this vector.

○ $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ √ $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}$

(3 pts) (d) Let $B = \begin{bmatrix} -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & 1 & 2 \\ 2 & 2 & 2 & -2 & 1 \end{bmatrix}$. Only one of the following statements is true. Select this statement.

○ $\text{rank}(B) = \text{rank}(A)$ √ $\text{Null}(B) = \text{Col}(A)$ ○ $\text{Col}(B) = \text{Null}(A)$ ○ $B^\intercal B = A^\intercal A$

○ B is the projection matrix to the left null space of A.

(3 pts) (e) Find X such that the projection matrix onto the row space of A is $X(X^\intercal X)^{-1}X^\intercal$. Clearly explain your reasoning to receive credit. Fill in the provided blank to make your answer clear.

Solution. Our desired X needs to be a matrix whose columns form a basis of the row space of A. As discussed in class, the nonzero rows of $R = \text{rref}(A)$ form a basis of the row space of A, so taking

$$X = \begin{bmatrix} 1 & 0 \\ 5 & 0 \\ 0 & 1 \\ 7 & 2 \\ 4 & 3 \\ 1 & 1 \\ 3 & 5 \end{bmatrix}$$

works.

(8 pts) **Problem 2.** Consider $\overset{P}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{A}{\begin{bmatrix} 0 & 0 & 5 & 3 \\ 3 & 7 & 5 & 2 \\ 9 & 21 & 25 & 12 \end{bmatrix}} = \overset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}} \overset{U}{\begin{bmatrix} 3 & 7 & 5 & 2 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$.

Use the algorithm discussed in class to calculate the matrices P, L, and U. Fill in the blank matrices above to make your answer clear. ***To receive points your work must be neatly organized and easy to follow.***

Solution. Following the algorithm from class, we have

$$\overset{A}{\begin{bmatrix} 0 & 0 & 5 & 3 \\ 3 & 7 & 5 & 2 \\ 9 & 21 & 25 & 12 \end{bmatrix}} \xrightarrow{r_1 \leftrightarrow r_2} \overset{U}{\begin{bmatrix} 3 & 7 & 5 & 2 \\ 0 & 0 & 5 & 3 \\ 9 & 21 & 25 & 12 \end{bmatrix}} \quad \overset{L}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}} \quad \overset{P}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$\xrightarrow{r_3 - 3 \cdot r_1 \to r_3} \begin{bmatrix} 3 & 7 & 5 & 2 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 10 & 6 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{r_3 - 2 \cdot r_2 \to r_3} \begin{bmatrix} 3 & 7 & 5 & 2 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 2 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(8 pts) **Problem 3.** The Gram-Schmidt algorithm can be used as an alternative to row-reducing to determine the pivot and nonpivot columns of a matrix. Find the vectors $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3\}$ obtained by applying the Gram-Schmidt algorithm to the columns of the matrix A depicted to the right of this paragraph. If any $\boldsymbol{w}_k$ is the zero vector, then do not use $\boldsymbol{w}_k$ to calculate the subsequent vectors. The kth column of A is a pivot column if and only if $\boldsymbol{w}_k \neq \boldsymbol{O}$!

Use this procedure to determine the pivot and nonpivot columns of A.

$$A = \begin{bmatrix} -1 & -4 & 5 \\ 1 & 4 & -5 \\ -1 & 2 & -7 \end{bmatrix}$$

Solution. The Gram-Schmidt algorithm gives

$$\begin{aligned}
\boldsymbol{w}_1 &= \begin{bmatrix} -1 & 1 & -1 \end{bmatrix}^\intercal \\
\boldsymbol{w}_2 &= \begin{bmatrix} -4 & 4 & 2 \end{bmatrix}^\intercal - \operatorname{proj}_{\boldsymbol{w}_1}(\begin{bmatrix} -4 & 4 & 2 \end{bmatrix}^\intercal) \\
&= \begin{bmatrix} -4 & 4 & 2 \end{bmatrix}^\intercal - \begin{bmatrix} -2 & 2 & -2 \end{bmatrix}^\intercal \\
&= \begin{bmatrix} -2 & 2 & 4 \end{bmatrix}^\intercal \\
\boldsymbol{w}_3 &= \begin{bmatrix} 5 & -5 & -7 \end{bmatrix}^\intercal - \operatorname{proj}_{\boldsymbol{w}_1}(\begin{bmatrix} 5 & -5 & -7 \end{bmatrix}^\intercal) - \operatorname{proj}_{\boldsymbol{w}_2}(\begin{bmatrix} 5 & -5 & -7 \end{bmatrix}^\intercal) \\
&= \begin{bmatrix} 5 & -5 & -7 \end{bmatrix}^\intercal - \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}^\intercal - \begin{bmatrix} 4 & -4 & -8 \end{bmatrix}^\intercal \\
&= \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^\intercal
\end{aligned}$$

This tells us that the pivot columns are the first two and the third is a nonpivot column.

Problem 4. Suppose that λ is an eigenvalue of a 2025×2025 matrix A.

(5 pts) (*a*) Show that λ is also an eigenvalue of $A^\intercal$. Your solution should read clearly as a single string of equalities.

Solution. We are given that λ is an eigenvalue of A, which means that $\operatorname{rank}(\lambda \cdot I_{2025} - A) < 2025$. We wish to demonstrate that λ is also an eigenvalue of $A^\intercal$, which means we wish to demonstrate that $\operatorname{rank}(\lambda \cdot I_{2025} - A^\intercal) < 2025$. Recalling that transposition does not impact rank, we then have

$$\operatorname{rank}(\lambda \cdot I_{2025} - A^\intercal) = \operatorname{rank}\left((\lambda \cdot I_{2025} - A^\intercal)^\intercal\right) = \operatorname{rank}\left(\lambda \cdot I_{2025}^\intercal - (A^\intercal)^\intercal\right) = \operatorname{rank}(\lambda \cdot I_{2025} - A) < 2025$$

This demonstrates that λ is indeed an eigenvalue of $A^\intercal$.

(2 pts) (*b*) Part (*a*) of this problem asserts that λ is an eigenvalue of both A and $A^\intercal$. All but one of the following statements is guaranteed to be true. Select the statement that is not guaranteed to be true.

◯ $\operatorname{gm}_A(\lambda) = \operatorname{gm}_{A^\intercal}(\lambda)$ ◯ $\dim \mathcal{E}_A(\lambda) = \dim \mathcal{E}_{A^\intercal}(\lambda)$ √ $\mathcal{E}_A(\lambda) = \mathcal{E}_{A^\intercal}(\lambda)$

◯ $\operatorname{rank}(\lambda \cdot I_{2025} - A) = \operatorname{rank}(\lambda \cdot I_{2025} - A^\intercal)$ ◯ $\operatorname{nullity}(\lambda \cdot I_{2025} - A) = \operatorname{nullity}(\lambda \cdot I_{2025} - A^\intercal)$

(6 pts) **Problem 5.** The scalar $\lambda = 2$ is an eigenvalue of $A = \begin{bmatrix} 15 & -13 & -55 & -13 \\ 3 & -1 & -11 & -3 \\ -1 & 1 & 7 & 1 \\ 12 & -12 & -56 & -10 \end{bmatrix}$ and it is known that $\operatorname{gm}_A(\lambda) = 2$. Us this information to find a basis of $\mathcal{E}_A(\lambda)$. Clearly explain your reasoning to receive credit.

Solution. We want a basis of $\mathcal{E}_A(\lambda) = \operatorname{Null}(\lambda \cdot I - A)$. From class we know that the dimension of this eigenspace is $\operatorname{gm}_A(\lambda) = 2$, so we need only find two independent vectors. To do so, note that

$$\mathcal{E}_A(\lambda) = \operatorname{Null} \overbrace{\begin{bmatrix} -13 & 13 & 55 & 13 \\ -3 & 3 & 11 & 3 \\ 1 & -1 & -5 & -1 \\ -12 & 12 & 56 & 12 \end{bmatrix}}^{\lambda \cdot I - A}$$

There are two easily-observable column relations

$$\operatorname{Col}_2 = -\operatorname{Col}_1 \qquad \operatorname{Col}_4 = -\operatorname{Col}_1$$

We may rewrite these column relations as

$$\operatorname{Col}_1 + \operatorname{Col}_2 = \boldsymbol{O} \qquad \operatorname{Col}_1 + \operatorname{Col}_4 = \boldsymbol{O}$$

This means that $\begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^\intercal, \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^\intercal \in \operatorname{Null}(\lambda \cdot I - A)$. Since these two vectors are independent, they therefore form a basis of $\mathcal{E}_A(\lambda)$.

Problem 6. The data below depicts a 4×7 matrix A and a 4×7 matrix B given by

$$A = \begin{bmatrix} 2 & -1 & 1 & 3 & -8 & -8 & 2 \\ -1 & 2 & 0 & -2 & 7 & 7 & -3 \\ 2 & -2 & 1 & 2 & -5 & -5 & 2 \\ -1 & -3 & -1 & 0 & -3 & -3 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 2 & -6 & 0 & -2 & -1 \\ 0 & 1 & 3 & -4 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & 2 & -6 & -1 & -1 & -1 \end{bmatrix}$$

It is known that $\operatorname{Col}(A^\intercal) = \operatorname{Null}(B)$ and that columns one, two, and five are the "pivot columns" of B (which implies $\operatorname{rank}(B) = 3$.

(4 pts) (*a*) Some, but not necessarily all, of the following lists of vectors is linearly independent. Select these lists (one point each).

○ All columns of A. √ All rows of A. √ The first two columns of A.

○ The first three columns of B.

(3 pts) (*b*) Of all the following vectors, it is only possible to express one as a linear combination of the rows of A. Select this vector.

○ $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ √ $\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ ○ $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

(4 pts) (*c*) Suppose we calculated $A = QR$. Then Q would be __4__ $\times$ __4__ and $\operatorname{trace}(QQ^\intercal) =$ __4__.

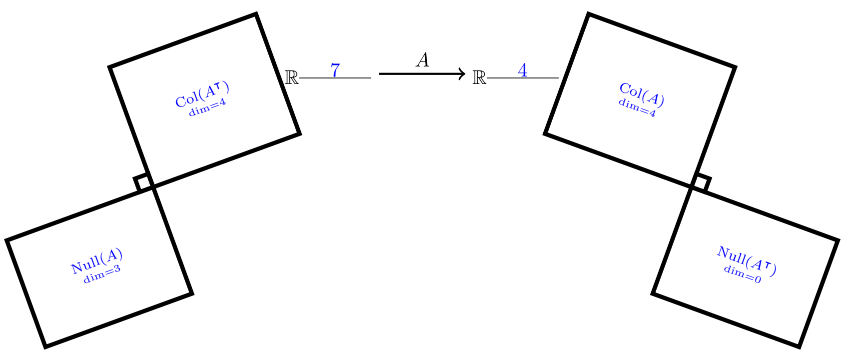
(4 pts) (*d*) Fill in each of the following blanks with a ">" sign, a "<" sign, or an "=" sign (one point each).

$\dim \operatorname{Null}(A)$ __<__ $\dim \operatorname{Null}(B)$ $\qquad$ $\dim \operatorname{Null}(A^\intercal)$ __<__ $\dim \operatorname{Null}(B^\intercal)$

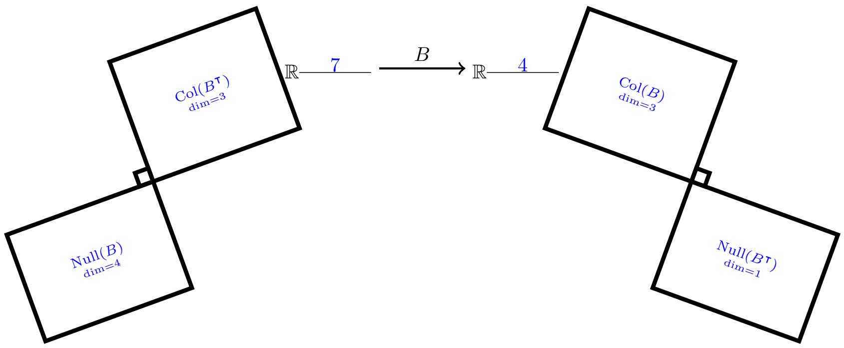
$\dim \operatorname{Col}(A)$ __>__ $\dim \operatorname{Col}(B)$ $\qquad$ $\dim \operatorname{Col}(A^\intercal)$ __>__ $\dim \operatorname{Col}(B^\intercal)$

Use the space below for any necessary scratch work.

Solution. The picture of the four fundamental subspaces of A is



The picture of the four fundamental subspaces of B is



(7 pts) **Problem 7.** Find a matrix A such that $\operatorname{Null}(A) = \operatorname{Span}\left\{\begin{bmatrix}1\\2\\1\\-1\end{bmatrix}, \begin{bmatrix}3\\7\\6\\8\end{bmatrix}, \begin{bmatrix}2\\1\\2\\-35\end{bmatrix}\right\}$. Clearly explain your reasoning to receive credit.

Solution. This is the same as asking for $\operatorname{Null}(A) = \operatorname{Col}\overset{B}{\begin{bmatrix}1 & 3 & 2\\ 2 & 7 & 1\\ 1 & 6 & 2\\ -1 & 8 & -35\end{bmatrix}}$. The criteria that any vector $\boldsymbol{b} = \begin{bmatrix}b_1 & b_2 & b_3 & b_4\end{bmatrix}^\intercal$ belongs to the column space of B is unveiled by the following row-reductions

$$\left[\begin{array}{rrr|r}1 & 3 & 2 & b_1\\ 2 & 7 & 1 & b_2\\ 1 & 6 & 2 & b_3\\ -1 & 8 & -35 & b_4\end{array}\right] \to \left[\begin{array}{rrr|r}1 & 3 & 2 & b_1\\ 0 & 1 & -3 & -2\,b_1+b_2\\ 0 & 3 & 0 & -b_1+b_3\\ 0 & 11 & -33 & b_1+b_4\end{array}\right] \to \left[\begin{array}{rrr|r}1 & 3 & 2 & b_1\\ 0 & 1 & -3 & -2\,b_1+b_2\\ 0 & 0 & 9 & 5\,b_1-3\,b_2+b_3\\ 0 & 0 & 0 & 23\,b_1-11\,b_2+b_4\end{array}\right]$$

Here, we realize that the consistency of our system is enforced by the single equation $23\,b_1 - 11\,b_2 + b_4 = 0$. This is the same as $\operatorname{Null}(A) = \operatorname{Col}(B)$ where $A = \begin{bmatrix}23 & -11 & 0 & 1\end{bmatrix}$.

(6 pts) **Problem 8.** The matrix R below is a complex matrix in reduced row echelon form.

$$R = \begin{bmatrix}1 & 0 & -i+2 & 0 & i\\ 0 & 1 & i+1 & 0 & -3i\\ 0 & 0 & 0 & 1 & -5i+7\end{bmatrix}$$

Calculate $\langle \boldsymbol{x}_1, \boldsymbol{x}_2\rangle$, where $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are the "pivot solutions" to $R\boldsymbol{x} = \boldsymbol{O}$. Clearly explain your reasoning to receive credit. Fill in the provided blank to make your answer clear.

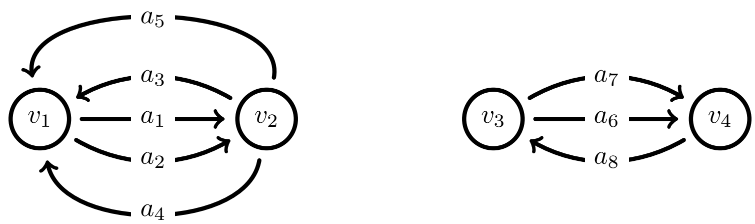
Solution. The system $R\boldsymbol{x} = \boldsymbol{O}$ has dependent variables x_1, x_2, x_4 and free variables $x_3 = c_1$ and $x_5 = c_2$. The general solution to $R\boldsymbol{x} = \boldsymbol{O}$ is then

$$\boldsymbol{x} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix} = \begin{bmatrix}-(-i+2)c_1 - ic_2\\ -(i+1)c_1 + 3\,ic_2\\ c_1\\ -(-5i+7)c_2\\ c_2\end{bmatrix} = c_1\overset{\boldsymbol{x}_1}{\begin{bmatrix}i-2\\ -i-1\\ 1\\ 0\\ 0\end{bmatrix}} + c_2\overset{\boldsymbol{x}_2}{\begin{bmatrix}-i\\ 3\,i\\ 0\\ 5\,i-7\\ 1\end{bmatrix}}$$

The inner product of the "pivot solutions" to $R\boldsymbol{x} = \boldsymbol{O}$ is then

$$\begin{aligned}\langle \boldsymbol{x}_1, \boldsymbol{x}_2\rangle &= \overline{(i-2)}\cdot(-i) + \overline{(-i-1)}\cdot(3\,i)\\ &= (-i-2)\cdot(-i) + (i-1)\cdot(3\,i)\\ &= -i^2 + 2\,i + 3\,i^2 - 3\,i\\ &= -1 + 2\,i - 3 - 3\,i\\ &= -i - 4\end{aligned}$$

Problem 9. Let A be the incidence matrix of the directed graph G given by



(4 pts) (*a*) The projection of $\begin{bmatrix}3 & -5 & 6 & -4\end{bmatrix}^\intercal$ onto the column space of A is $\begin{bmatrix}4 & -4 & 5 & -5\end{bmatrix}^\intercal$. Use this information to find the error E in solving the least squares system associated to $A\boldsymbol{x} = \begin{bmatrix}3 & -5 & 6 & -4\end{bmatrix}^\intercal$. Clearly explain your reasoning to receive credit. Fill in the blank below to make your answer clear.

Solution. The *error* in least squares is $\|\boldsymbol{b} - P\boldsymbol{b}\|^2$ where P is projection to the column space of A. We're given this data directly, so we calculate

$$E = \left\| \begin{bmatrix}3\\-5\\6\\-4\end{bmatrix} - \begin{bmatrix}4\\-4\\5\\-5\end{bmatrix} \right\|^2 = \left\| \begin{bmatrix}-1\\-1\\1\\1\end{bmatrix} \right\|^2 = 4$$

(7 pts) (*b*) Find the error E in solving the least squares system associated to $A\boldsymbol{x} = \begin{bmatrix}7 & -1 & -4 & -2\end{bmatrix}^\intercal$. Clearly explain your reasoning to receive credit. Fill in the blank below to make your answer clear.

Solution. The *error* in least squares is $\|\boldsymbol{b} - P\boldsymbol{b}\|^2$ where P is projection to the column space of A. Note, however, that $\boldsymbol{b} - P\boldsymbol{b}$ is the projection of $\boldsymbol{b}$ to $\operatorname{Null}(A^\intercal)$.

To find the projection matrix to the left null space of A, note that this directed graph has two connected components where the first two nodes cluster together and the last two nodes cluster together. Our technique for finding a basis from class uses the vectors $\boldsymbol{l}_1 = \begin{bmatrix}1 & 1 & 0 & 0\end{bmatrix}^\intercal$ and $\boldsymbol{l}_2 = \begin{bmatrix}0 & 0 & 1 & 1\end{bmatrix}^\intercal$. Assembling these vectors into an "X" matrix yields

$$X = \begin{bmatrix}1 & 0\\1 & 0\\0 & 1\\0 & 1\end{bmatrix} \qquad X^\intercal X = \begin{bmatrix}2 & 0\\0 & 2\end{bmatrix} \qquad (X^\intercal X)^{-1} = \frac{1}{4}\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}$$

The projection of $\begin{bmatrix}7 & -1 & -4 & -2\end{bmatrix}^\intercal$ to the left null space of A is then

$$\overset{X}{\begin{bmatrix}1 & 0\\1 & 0\\0 & 1\\0 & 1\end{bmatrix}} \overset{(X^\intercal X)^{-1}}{\frac{1}{4}\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}} \overset{X^\intercal}{\begin{bmatrix}1 & 1 & 0 & 0\\0 & 0 & 1 & 1\end{bmatrix}} \begin{bmatrix}7\\-1\\-4\\-2\end{bmatrix} = \begin{bmatrix}3\\3\\-3\\-3\end{bmatrix}$$

Our desired error is the square length of this vector, which is $E = 36$.

(8 pts) **Problem 10.** The following system of linear equations has *exactly one solution.*

$$x_1 - 5\,x_2 - 4\,x_3 = 1 \qquad x_1 - 2\,x_2 + x_3 = 3 \qquad -x_1 + 4\,x_2 - x_3 = 3$$

Find the value of x_2 in this system *without row-reducing any augmented matrices.* Clearly explain your reasoning to receive credit and check your work closely for numerical accuracy (the tools we've developed in the course allow for this problem to be solved by hand without numerical error).

Solution. This is $A\boldsymbol{x} = \boldsymbol{b}$ where

$$A = \begin{bmatrix} 1 & -5 & -4 \\ 1 & -2 & 1 \\ -1 & 4 & -1 \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$$

We're asked to isolate a variable in a linear system without row-reducing any augmented matrices, so we suspect Cramer's Rule to be relevant here.

The determinant of A is

$$\overset{A}{\begin{vmatrix} 1 & -5 & -4 \\ 1 & -2 & 1 \\ -1 & 4 & -1 \end{vmatrix}} \xlongequal{\substack{\boldsymbol{r}_2 - \boldsymbol{r}_1 \to \boldsymbol{r}_2 \\ \boldsymbol{r}_3 + \boldsymbol{r}_1 \to \boldsymbol{r}_3}} \begin{vmatrix} 1 & -5 & -4 \\ 0 & 3 & 5 \\ 0 & -1 & -5 \end{vmatrix} \xlongequal{\text{Col}_1} \begin{vmatrix} 3 & 5 \\ -1 & -5 \end{vmatrix} = -10$$

Cramer's Rule then asserts that the next relevant calculation in finding x_2 is

$$\overset{A_1}{\begin{vmatrix} 1 & 1 & -4 \\ 1 & 3 & 1 \\ -1 & 3 & -1 \end{vmatrix}} \xlongequal{\substack{\boldsymbol{r}_2 - \boldsymbol{r}_1 \to \boldsymbol{r}_2 \\ \boldsymbol{r}_3 + \boldsymbol{r}_1 \to \boldsymbol{r}_3}} \begin{vmatrix} 1 & 1 & -4 \\ 0 & 2 & 5 \\ 0 & 4 & -5 \end{vmatrix} \xlongequal{\boldsymbol{r}_3 - 2\cdot\boldsymbol{r}_1 \to \boldsymbol{r}_3} \begin{vmatrix} 1 & 1 & -4 \\ 0 & 2 & 5 \\ 0 & 0 & -15 \end{vmatrix} = -30$$

According to Cramer's Rule, the value of x_2 in this system is $x_2 = \dfrac{\det(A_1)}{\det(A)} = \dfrac{-30}{-10} = 3.$

(8 pts) **Problem 11.** Find all $t \in \mathbb{C}$ such that $A = \begin{bmatrix} t-3 & -9 & -5 \\ -1 & t-3 & -2 \\ 4 & 12 & t+7 \end{bmatrix}$ is *singular*. Clearly explain your reasoning to receive credit.

Solution. This is the same as asking for every $t \in \mathbb{C}$ such that $\det(A) = 0$. The determinant of A is

$$\begin{aligned}
\begin{vmatrix} t-3 & -9 & -5 \\ -1 & t-3 & -2 \\ 4 & 12 & t+7 \end{vmatrix} &\xlongequal{\boldsymbol{r}_3 + 4\cdot \boldsymbol{r}_2 \to \boldsymbol{r}_3} \begin{vmatrix} t-3 & -9 & -5 \\ -1 & t-3 & -2 \\ 0 & 4t & t-1 \end{vmatrix} \\
&\xlongequal{\text{Col}_1} (t-3)\begin{vmatrix} t-3 & -2 \\ 4t & t-1 \end{vmatrix} - (-1)\begin{vmatrix} -9 & -5 \\ 4t & t-1 \end{vmatrix} \\
&= (t-3)\{(t-3)(t-1) - (-2)(4t)\} + (-9)(t-1) - (4t)(-5) \\
&= (t-3)\{t^2 - 4t + 3 + 8t\} - 9t + 9 + 20t \\
&= (t-3)\{t^2 + 4t + 3\} + 11t + 9 \\
&= t^3 + 4t^2 + 3t - 3t^2 - 12t - 9 + 11t + 9 \\
&= t^3 + t^2 + 2t \\
&= t(t^2 + t + 2)
\end{aligned}$$

This is a monic polynomial of degree three. Our work above demonstrates that one of the roots is $t = 0$. The other two are unveiled by the quadratic formula

$$t = \frac{-1 \pm \sqrt{(-1)^2 - 4(1)(2)}}{2} = \frac{-1 \pm \sqrt{1-8}}{2} = \frac{-1 \pm \sqrt{-7}}{2} = \frac{-1 \pm \sqrt{7}\,i}{2}$$

The values of $t \in \mathbb{C}$ that make A singular are thus

$$t = 0 \qquad t = \frac{-1 - \sqrt{7}\,i}{2} \qquad t = \frac{-1 + \sqrt{7}\,i}{2}$$