Duke University

Math 218D-2

Matrices and Vectors

Exam II

Name: ______________________ *Unique ID:* ______________________

I have adhered to the Duke Community Standard in completing this exam.

Signature: ______________________

June 17, 2025

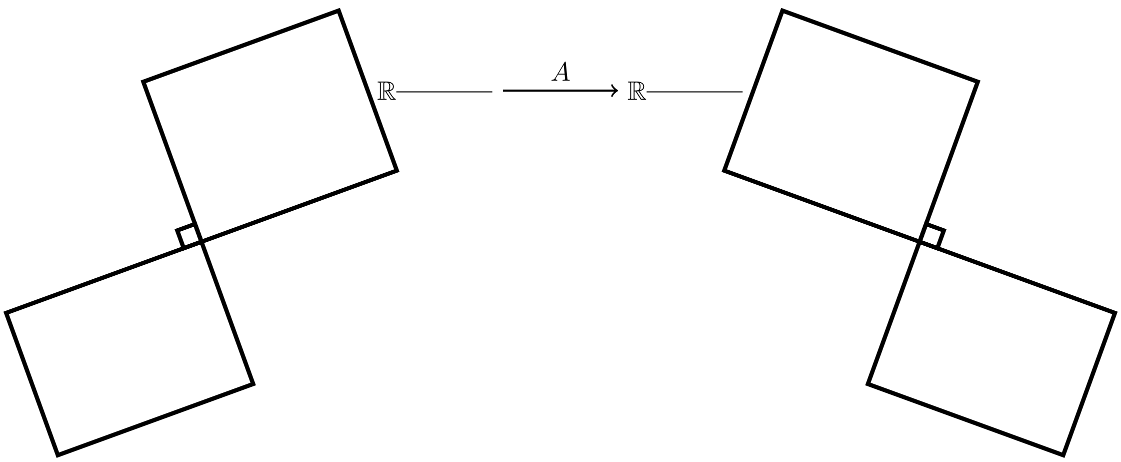
- There are 100 points and 11 problems on this 100-minute exam.
- Unless otherwise stated, your answers must be supported by clear and coherent work to receive credit.
- The back of each page of this exam is left blank and may be used for scratch work.
- Scratch work will not be graded unless it is clearly labeled and requested in the body of the original problem.

Problem 1. Consider the following $EA = R$ factorization.

$$\overset{E}{\begin{bmatrix} 0 & 1 & 2 & 0 & 0 \\ -3 & 1 & -2 & 0 & 1 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & 1 & 2 \\ 2 & 2 & 2 & -2 & 1 \end{bmatrix}} \overset{A}{\begin{bmatrix} 13 & 65 & 5 & 101 & 67 & * & 64 \\ 25 & 125 & 10 & 195 & 130 & * & 125 \\ -12 & -60 & -5 & -94 & -63 & * & -61 \\ 21 & 105 & 8 & 163 & 108 & * & 103 \\ -10 & -50 & -4 & -78 & -52 & * & -50 \end{bmatrix}} = \overset{R}{\begin{bmatrix} 1 & 5 & 0 & 7 & 4 & 1 & 3 \\ 0 & 0 & 1 & 2 & 3 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}$$

Note that the data in the sixth column of A is missing and marked $*$.

(6 pts) (*a*) Fill in every missing label in the picture of the four fundamental subspaces of A below, including the dimension of each fundamental subspace.



(2 pts) (*b*) The missing column of A is $\begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix}$

(2 pts) (*c*) Only one of the following vectors belongs to the column space of A. Select this vector.

○ $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}$

(3 pts) (*d*) Let $B = \begin{bmatrix} -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & 1 & 2 \\ 2 & 2 & 2 & -2 & 1 \end{bmatrix}$. Only one of the following statements is true. Select this statement.

○ $\operatorname{rank}(B) = \operatorname{rank}(A)$ ○ $\operatorname{Null}(B) = \operatorname{Col}(A)$ ○ $\operatorname{Col}(B) = \operatorname{Null}(A)$ ○ $B^\intercal B = A^\intercal A$

○ B is the projection matrix to the left null space of A.

(3 pts) (*e*) Find X such that the projection matrix onto the row space of A is $X(X^\intercal X)^{-1}X^\intercal$. Clearly explain your reasoning to receive credit. Fill in the provided blank to make your answer clear.

$$X = \begin{bmatrix} \\ \\ \\ \\ \\ \\ \\ \end{bmatrix}$$

(8 pts) **Problem 2.** Consider

$$\overset{P}{\begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}} \overset{A}{\begin{bmatrix} 0 & 0 & 5 & 3 \\ 3 & 7 & 5 & 2 \\ 9 & 21 & 25 & 12 \end{bmatrix}} = \overset{L}{\begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}} \overset{U}{\begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \end{bmatrix}}.$$

Use the algorithm discussed in class to calculate the matrices P, L, and U. Fill in the blank matrices above to make your answer clear. ***To receive points your work must be neatly organized and easy to follow.***

(8 pts) **Problem 3.** The Gram-Schmidt algorithm can be used as an alternative to row-reducing to determine the pivot and nonpivot columns of a matrix. Find the vectors $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3\}$ obtained by applying the Gram-Schmidt algorithm to the columns of the matrix A depicted to the right of this paragraph. If any $\boldsymbol{w}_k$ is the zero vector, then do not use $\boldsymbol{w}_k$ to calculate the subsequent vectors. The kth column of A is a pivot column if and only if $\boldsymbol{w}_k \neq \boldsymbol{O}$!
Use this procedure to determine the pivot and nonpivot columns of A.

$$A = \begin{bmatrix} -1 & -4 & 5 \\ 1 & 4 & -5 \\ -1 & 2 & -7 \end{bmatrix}$$

Problem 4. Suppose that λ is an eigenvalue of a 2025×2025 matrix A.

(5 pts) (a) Show that λ is also an eigenvalue of $A^\intercal$. Your solution should read clearly as a single string of equalities.

(2 pts) (b) Part (a) of this problem asserts that λ is an eigenvalue of both A and $A^\intercal$. All but one of the following statements is guaranteed to be true. Select the statement that is not guaranteed to be true.

○ $\operatorname{gm}_A(\lambda) = \operatorname{gm}_{A^\intercal}(\lambda)$ ○ $\dim \mathcal{E}_A(\lambda) = \dim \mathcal{E}_{A^\intercal}(\lambda)$ ○ $\mathcal{E}_A(\lambda) = \mathcal{E}_{A^\intercal}(\lambda)$

○ $\operatorname{rank}(\lambda \cdot I_{2025} - A) = \operatorname{rank}(\lambda \cdot I_{2025} - A^\intercal)$ ○ $\operatorname{nullity}(\lambda \cdot I_{2025} - A) = \operatorname{nullity}(\lambda \cdot I_{2025} - A^\intercal)$

(6 pts) **Problem 5.** The scalar $\lambda = 2$ is an eigenvalue of $A = \begin{bmatrix} 15 & -13 & -55 & -13 \\ 3 & -1 & -11 & -3 \\ -1 & 1 & 7 & 1 \\ 12 & -12 & -56 & -10 \end{bmatrix}$ and it is known that $\operatorname{gm}_A(\lambda) = 2$. Us this information to find a basis of $\mathcal{E}_A(\lambda)$. Clearly explain your reasoning to receive credit.

Problem 6. The data below depicts a 4×7 matrix A and a 4×7 matrix B given by

$$A = \begin{bmatrix} 2 & -1 & 1 & 3 & -8 & -8 & 2 \\ -1 & 2 & 0 & -2 & 7 & 7 & -3 \\ 2 & -2 & 1 & 2 & -5 & -5 & 2 \\ -1 & -3 & -1 & 0 & -3 & -3 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 2 & -6 & 0 & -2 & -1 \\ 0 & 1 & 3 & -4 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & 2 & -6 & -1 & -1 & -1 \end{bmatrix}$$

It is known that $\operatorname{Col}(A^\intercal) = \operatorname{Null}(B)$ and that columns one, two, and five are the "pivot columns" of B (which implies $\operatorname{rank}(B) = 3$.

(4 pts) (*a*) Some, but not necessarily all, of the following lists of vectors is linearly independent. Select these lists (one point each).

○ All columns of A. ○ All rows of A. ○ The first two columns of A.

○ The first three columns of B.

(3 pts) (*b*) Of all the following vectors, it is only possible to express one as a linear combination of the rows of A. Select this vector.

○ $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ ○ $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

(4 pts) (*c*) Suppose we calculated $A = QR$. Then Q would be ______ $\times$ ______ and $\operatorname{trace}(QQ^\intercal) =$ ______.

(4 pts) (*d*) Fill in each of the following blanks with a ">" sign, a "<" sign, or an "=" sign (one point each).

$\dim \operatorname{Null}(A)$ ______ $\dim \operatorname{Null}(B)$ $\qquad$ $\dim \operatorname{Null}(A^\intercal)$ ______ $\dim \operatorname{Null}(B^\intercal)$

$\dim \operatorname{Col}(A)$ ______ $\dim \operatorname{Col}(B)$ $\qquad$ $\dim \operatorname{Col}(A^\intercal)$ ______ $\dim \operatorname{Col}(B^\intercal)$

Use the space below for any necessary scratch work.

(7 pts) **Problem 7.** Find a matrix A such that $\text{Null}(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 6 \\ 8 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ -35 \end{bmatrix} \right\}$. Clearly explain your reasoning to receive credit.

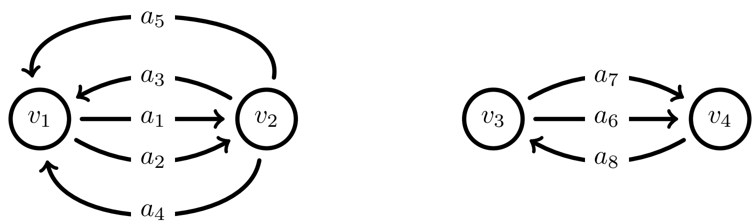
(6 pts) **Problem 8.** The matrix R below is a complex matrix in reduced row echelon form.

$$R = \begin{bmatrix} 1 & 0 & -i+2 & 0 & i \\ 0 & 1 & i+1 & 0 & -3i \\ 0 & 0 & 0 & 1 & -5i+7 \end{bmatrix}$$

Calculate $\langle \boldsymbol{x}_1, \boldsymbol{x}_2 \rangle$, where $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are the "pivot solutions" to $R\boldsymbol{x} = \boldsymbol{O}$. Clearly explain your reasoning to receive credit. Fill in the provided blank to make your answer clear.

$\langle \boldsymbol{x}_1, \boldsymbol{x}_2 \rangle =$ ______________

Problem 9. Let A be the incidence matrix of the directed graph G given by



(4 pts) (a) The projection of $\begin{bmatrix}3 & -5 & 6 & -4\end{bmatrix}^\intercal$ onto the column space of A is $\begin{bmatrix}4 & -4 & 5 & -5\end{bmatrix}^\intercal$. Use this information to find the error E in solving the least squares system associated to $A\boldsymbol{x} = \begin{bmatrix}3 & -5 & 6 & -4\end{bmatrix}^\intercal$. Clearly explain your reasoning to receive credit. Fill in the blank below to make your answer clear.

$E =$ ________

(7 pts) Find the error E in solving the least squares system associated to $A\boldsymbol{x} = \begin{bmatrix}7 & -1 & -4 & -2\end{bmatrix}^\intercal$. Clearly explain your reasoning to receive credit. Fill in the blank below to make your answer clear.

$E =$ ________

(8 pts)**Problem 10.** The following system of linear equations has *exactly one solution.*

$$x_1 - 5\,x_2 - 4\,x_3 = 1 \qquad x_1 - 2\,x_2 + x_3 = 3 \qquad -x_1 + 4\,x_2 - x_3 = 3$$

Find the value of x_2 in this system *without row-reducing any augmented matrices.* Clearly explain your reasoning to receive credit and check your work closely for numerical accuracy (the tools we've developed in the course allow for this problem to be solved by hand without numerical error).

(8 pts)**Problem 11.** Find all $t \in \mathbb{C}$ such that $A = \begin{bmatrix} t-3 & -9 & -5 \\ -1 & t-3 & -2 \\ 4 & 12 & t+7 \end{bmatrix}$ is *singular*. Clearly explain your reasoning to receive credit.